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Improper Colourings of Unit Disk Graphs [1]

F. Havet [a,2,4], R. J. Kang [b,3,5] and J.-S. Sereni [a,2,6]

[a] *MASCOTTE, I3S-CNRS/INRIA/UNSA, 2004 Route des Lucioles, BP93, F-06902, Sophia-Antipolis, France*

[b] *Oxford University, Department of Statistics, 1 South Parks Road, Oxford OX1 3TG, United Kingdom*

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1 Introduction

We investigate the following problem proposed by Alcatel. A satellite sends information to receivers on earth, each of which is listening on a chosen frequency. Technically, it is impossible for the satellite to precisely focus its signal onto a receiver. Part of the signal will be spread in an area around its destination and this creates noise for nearby receivers on the same frequency. However, a receiver is able to distinguish its signal if the sum of the noise does not become too large, i.e. does not exceed a certain threshold T. The problem

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[2] These two authors are partially supported by the European project FET-CRESCCO.

[3] This author is partially supported by NSERC of Canada and the Commonwealth Scholarship Commission (UK).

[4] Email: `fhavet@sophia.inria.fr`

[5] Email: `kang@stats.ox.ac.uk`

[6] Email: `sereni@sophia.inria.fr`

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is to assign frequencies to the receivers so that each receiver can distinguish its signal. We investigate this problem in the fundamental case where the noise area at a receiver does not depend on the frequency and where the "noise relation" is symmetric; that is, if a receiver u is in the noise area of a receiver v, then v is in the noise area of u. Moreover, the intensity I of the noise created by a signal is independent of the frequency and the receiver; hence, to distinguish its signal from the noise, a receiver must be in the noise area of at most $k = \lfloor \frac{T}{I} \rfloor$ receivers getting signals on the same frequency.

We model this problem as a graph colouring problem. Define a *noise graph* as follows: the vertices are the receivers and we put an edge between u and v if u is in the noise area of v. The frequencies are represented by colours; thus, an assignment of frequencies to receivers is equivalent to a k-improper colouring of the noise graph. The *impropriety* of a vertex v of a graph G under the colouring c, denoted by $\mathrm{im}^c_G(v)$, is the number of neighbours of v coloured $c(v)$. A colouring c is *k-improper* if all the vertices have impropriety at most k under c. Note that 0-improper colouring is the usual notion of proper colouring.

Improper colouring has been widely studied, particularly in the case of planar graphs. It is known that every planar graph is 0-improper 4-colourable (due to the 4-colour theorem [1]) and 2-improper 3-colourable [4,5,9]. In [3], the authors considered complexity issues of improper colourings and showed that the (general) k-improper l-colourability problem is NP-complete except for when $k = 0$ and $l = 2$ (or $l = 1$). They also proved that the problem remains intractable for planar graphs if $l = 2$ and $k \geq 1$ or $l = 3$ and $k = 1$.

In our case, we assume that the noise areas are represented by equal-sized disks in the plane, i.e. the noise graphs are unit disk graphs. Proper colouring (i.e. $k = 0$) for unit disk graphs and the subclass of weighted induced subgraphs of the triangular lattice has been widely studied due to its relation to the frequency assignment problem [7,2,6,8]. We extend this work by addressing two complexity issues. The first one concerns k-improper l-colourability of unit disk graphs and we show NP-completeness in all possible cases. The second concerns improper multi-colourings of induced subgraphs of the triangular lattice, for which we also prove NP-completeness results.

2 Improper colourings of unit disk graphs

Recall the definition of a unit disk graph. We are given an arbitrary set of n points fixed in the plane and a fixed positive quantity d. At each point, we centre a disk of diameter d. We connect two points if their disks' interiors intersect; that is, we connect two points if they are less than distance d apart. Any graph isomorphic to a graph that is constructed this way is called a *unit disk graph*. For any unit disk graph G, a set of points in the plane that, together with some value of d, gives rise to G is called a *representation* of G.

Theorem 2.1 *For fixed $k \geq 1$, the following problem is NP-complete:*
INSTANCE: a unit disk graph G (with a representation).
QUESTION: is there a k-improper 2-colouring of G?

Clark, Colbourn and Johnson [2] showed that it is NP-complete to determine if a unit disk graph is 3-colourable. By using a fairly straightforward substitution (of edges with a specially constructed sequence of unit disks), they managed to reduce 3-colourability of planar graphs with maximum degree 4 to 3-colourability of unit disk graphs. Their proof relies on a special embedding of planar graphs with maximum degree at most 4 in which the arcs of the embedding lie only on lines of the integer grid.

In the proof of Theorem 2.1, we mimic the approach in [2]. Our reduction, though, is from k-improper 2-colourability of planar graphs to k-improper 2-colourability of unit disk graphs, and this gives us two further considerations. First, there is no constraint on the maximum degree of the given planar graphs; thus, we must use a different kind of planar embedding, a so-called *box-orthogonal embedding*, in which the arcs of the embedding lie on lines of the integer grid but each vertex is represented by a rectangle. Second, the auxiliary graphs which are substituted into this embedding must not only transmit colourability but also impropriety information.

Theorem 2.2 *For fixed $k \geq 0$ and $l \geq 3$, the following problem is NP-complete:*
INSTANCE: a unit disk graph G (with a representation).
QUESTION: is there a k-improper l-colouring of G?

Gräf, Stumpf and Weißenfels [6] extended the result of Clark et al by showing that it is NP-complete to determine if a unit disk graph is l-colourable, for any fixed $l \geq 3$. Because l-colourability of planar graphs is not NP-complete for $l \geq 4$ (due to the 4-colour theorem), they used a reduction from general l-colourability. This approach gave two new problems. First, there is no constraint on the maximum degree of the given graphs; second, there are edge-crossings to cope with. They developed a special rectilinear embedding of general graphs, as well as an l-crossing auxiliary graph to handle these issues.

To prove Theorem 2.2, we generalise the approach of [6] to k-improper

l-colourability. We use the same embedding and most of the same auxiliary graphs (up to replacement of each vertex by a $(k+1)$-clique). The most significant issue is that we must use a crossing auxiliary graph that is significantly larger than that of Gräf et al.

Each unit interval graph is clearly a unit disk graph. It is not difficult to prove that the k-improper l-colourability problem restricted to (general) interval graphs is in P for fixed k and l. If I is a unit interval graph of clique number ω, then its k-improper chromatic number (i.e. the least integer l for which it is k-improper l-colourable) is either $\lceil \frac{\omega}{k+1} \rceil$ or $\lceil \frac{\omega}{k+1} \rceil + 1$; however, we do not know yet if it is polynomial to decide between these values.

3 Weighted improper colourings in the triangular lattice

Given a graph G, a weight vector ω for G is a nonzero vector of nonnegative integers indexed by the vertices of G. Colouring a weighted graph (G, ω) means assigning to each vertex v a list of ω_v colours. In this section, we will investigate two kinds of improper multi-colourings and apply them in the case of weighted induced subgraphs of the triangular lattice. Recall that the triangular lattice T consists of points determined by all linear combinations of vectors $\boldsymbol{a} = (1, 0)$ and $\boldsymbol{b} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

Here is the first problem we consider: we take the most straightforward notion of k-improper colouring for weighted graphs (where we just consider usual k-improper colouring and apply it to the graph in which each vertex v is replaced by a clique of size ω_v) and, given a weighted induced subgraph (F, ω) of T, we want to determine if there is a 3-colouring which is k-improper. Formally, let c be a colouring of a weighted graph (G, ω). For any vertex $v \in G$ and any colour $x \in c(v)$, the *impropriety* of v with respect to x is:

$$\mathrm{im}^x(v) = \mathrm{mult}(x, c(v)) - 1 + \sum_{w \in N(v)} \mathrm{mult}(x, c(w))$$

where $\mathrm{mult}(x, L)$ denotes the number of times the colour x appears in the list L. A colouring c of (G, ω) is said to be *k-improper* if the impropriety of each vertex with respect to any colour applied to that vertex is at most k.

Theorem 3.1 *For any $k \geq 0$, the following problem is NP-complete:*
INSTANCE: a weighted induced subgraph (F, ω) of the triangular lattice.
QUESTION: is there a k-improper 3-colouring of (F, ω)?

The proof generalises the proof of McDiarmid and Reed [8], and the reduction is from 3-colourability of planar graphs with maximum degree 4.

The second problem we consider is the following: we take an alternative notion of improper colouring and, again, we want to find if there is a 3-colouring of a given weighted induced subgraph (F, ω) of T. Given a k-improper colouring c of a weighted graph (G, ω), we say that c is *distinct* if, for any vertex $v \in G$, the colours in $c(v)$ are all distinct. First of all, note that, if it is possible to find a distinct k-improper 3-colouring of a weighted induced subgraph of T, then the weight of any vertex may not exceed 3. If $k = 0$, the problem is NP-complete by Theorem 3.1, and the reader can refer to [8]. If k is at least 6, then, as the maximum degree of the triangular lattice is 6, the problem is trivial. We prove that the problem stays intractable up until $k = 6$.

Theorem 3.2 *For $0 \leq k \leq 5$, the following problem is NP-complete:*
INSTANCE: a weighted induced subgraph (F, ω) of the triangular lattice.
QUESTION: is there a distinct k-improper 3-colouring of (F, ω)?

Again, the reduction is to 3-colourability of planar graphs with maximum degree 4.

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