Channel Assignment and Improper Choosability of Graphs

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Abstract. We model a problem proposed by Alcatel, a satellite building company, using improper colourings of graphs. The relation between improper colourings and maximum average degree is underlined, which contributes to generalise and improve previous known results about improper colourings of planar graphs.

1 Introduction

In this paper, we investigate the following problem proposed by Alcatel, a satellite building company. A satellite sends information to receivers on earth, each of which is listening on a frequency. Technically it is impossible to focus the signal sent by the satellite exactly on receiver. So part of the signal is spread in an area around it creating noise for the other receivers displayed in this area and listening on the same frequency. A receiver is able to distinguish the signal directed to it from the extraneous noises it picks up if the sum of the noises does not become too big, i.e. does not exceed a certain threshold T. The problem is to assign frequency to the receivers in such a way that each receiver gets its dedicated signal properly. We investigate this problem in the fundamental case where the noise area at a receiver does not depend on the frequency and where the "noise relation" is symmetric that is if a receiver u is in the noise area of a receiver v then v is in the noise area of u. Moreover the intensity I of the noise created by a signal is independent of the frequency and the receiver. Hence to distinguish its signal from noises, a receiver must be in the noise area of at most $k = \left\lfloor \frac{T}{I} \right\rfloor$ receivers listening signals on the same frequency.

We model this problem in a graph colouring problem. We define a *noise graph*: the vertices are the receivers and we put an edge between u and v if u is in the noise area of v (and v in the noise area of u). The frequencies are represented by colours. So assigning frequencies to receivers is equivalent to k-improper colouring the noise graph. Indeed the *impropriety* of a vertex v of a graph G under the colouring c, denoted by $\mathrm{im}_G^c(v)$, is the number of neighbours of v coloured $c(v)$. A colouring is k*-improper* if all the vertices have impropriety at most k under it. Note that 0-improper colouring is the usual notion of proper colouring.

Due to some practical reasons (as, for instance, the specific environment of a receiver), the colour of each vertex v must be chosen among a list of colours

D. Kratsch (Ed.): WG 2005, LNCS 3787, pp. 81–90, 2005.

$L(v)$ (that represents the frequencies allowed for that receiver). Formally, given a graph G, an *l-list-assignment* L of G is an function which assigns to each vertex of G a list of at least l colours. An *L-colouring* of G is a vertex colouring in which each vertex v is assigned a colour of the list $L(v)$. G is *k-improper L-colourable* if there exists a k-improper L-colouring of G. G is said to be *k-improper l-choosable* if such a colouring exists for any l-list-assignment.

Improper choosability of planar graphs has been widely studied. In particular, any planar graph is known to be 0-improper 5-choosable [8] and 2-improper 3-choosable [3,6]. It is conjectured that any planar graph is 1-improper 4-choosable. Škrekovski [7] studied k-improper 2-choosability of planar graphs in relation with their girth (the *girth* of a graph G is the size of a smallest cycle of G). Denoting by g_k the smallest integer such that every planar graph of girth at least g_k is k-improper 2-choosable, he proved $6 \leq g_1 \leq 9$, $5 \leq g_2 \leq 7$, $5 \leq g_3 \leq 6$ and $\forall k \geq 4, g_k = 5$. In this paper, we study improper colourings of (not necessarily planar) graphs in relation with their density. Not only does this approach generalise and improve the results of [7] concerning planar graphs, but it also has practical interest since the noise graphs modelling Alcatel's networks have bounded density.

The *average degree* of a graph G, denoted by $\mathrm{Ad}(G)$, is the sum of the degree of each vertex divided by the number of vertices. The *maximum average degree* of G, denoted by $\mathrm{Mad}(G)$, is the maximum of the average degree of each of its subgraphs (including G). If G is not a forest, the *heart* of G, denoted by $h(G)$, is the biggest subgraph of G in which every vertex has degree at least 2. It can be obtained by consecutive removing of vertices of degree 1.

Proposition 1. *If G is not a forest, then* $\mathrm{Mad}(G) = \mathrm{Mad}(h(G))$.

Proof. As $h(G)$ is a subgraph of G, $\mathrm{Mad}(G) \geq \mathrm{Mad}(h(G))$. Let H be a subgraph of G such that $\mathrm{Mad}(G) = \mathrm{Ad}(H)$. Then H is not a forest since otherwise we would have $\mathrm{Mad}(G) < 2$ and G would be a forest. So $h(H)$ is defined and it is a subgraph of $h(G)$. Moreover, $h(H)$ has minimum degree at least 2, so adding to it vertices of degree 1 cannot increase its average degree: let H' be a supergraph obtained from $h(H)$ by adding $k \geq 1$ vertices of degree 1. We assume that $h(H)$ has n vertices. Then

$$\mathrm{Ad}(H') = \frac{n \times \mathrm{Ad}(h(H)) + 2k}{n+k} = \mathrm{Ad}(h(H)) + \frac{2k - k \times \mathrm{Ad}(h(H))}{n+k} \leq \mathrm{Ad}(h(H))$$

since $\mathrm{Ad}(h(H)) \geq 2$. So $\mathrm{Mad}(h(G)) \geq \mathrm{Ad}(h(H)) \geq \mathrm{Ad}(H) = \mathrm{Mad}(G)$.

Let $M(k,l)$ be the greatest real such that every graph of maximum average degree less than $M(k,l)$ is k-improper l-choosable. Obviously, $M(k_1,l) \leq M(k_2,l)$ if $k_1 \leq k_2$. We have that $M(k,1) = \frac{2k+2}{k+2}$ since a graph is k-improper 1-choosable if and only if it has maximum degree at most k (and a graph of maximum degree at least $k+1$ contains the star S_{k+1} as a subgraph, so it has maximum average degree at least $\frac{2k+2}{k+2}$). If $l \geq 2$, first note that any tree is 0-improper 2-choosable. Furthermore, for any $k \geq 0$, a graph G which is not a forest is k-improper 2-choosable if and only if its heart is. Hence, we shall restrict the study to graphs with minimum degree at least 2.

In the following section, we show:

Theorem 1. *For all $k \geq 0$, all graphs of maximum average degree less than $\frac{4k+4}{k+2}$ are k-improper 2-choosable.*

Theorem 2. *For all $k \geq 1$, $M(k,2) \leq \dfrac{4k^2+6k+4}{k^2+2k+2} = 4 - \dfrac{2k+4}{k^2+2k+2}$.*

We then generalise Theorem 1:

Theorem 3. *For all $l \geq 2$ and all $k \geq 0$, all graphs of maximum average degree less than $\frac{l(l+2k)}{l+k}$ are k-improper l-choosable.*

Corollary 1. *For any fixed l, $\lim\limits_{k \to +\infty} M(k,l) = 2l$.*

Using Euler's formula, one can show that if G is a planar graph with minimum degree at least 2 and girth at least g, then $\mathrm{Mad}(G) < \frac{2g}{g-2}$. So Theorem 1 immediately implies:

Corollary 2. *Let G be a planar graph of girth g.*

1. *If $g \geq 8$ then G is 1-improper 2-choosable, so $g_1 \leq 8$.*
2. *If $g \geq 6$ then G is 2-improper 2-choosable, so $g_3 \leq g_2 \leq 6$.*
3. *If $g \geq 5$ then G is 4-improper 2-choosable, so $g_k \leq 5$ for $k \geq 4$.*

Some proofs are omitted or just sketched. The detailed proofs are presented in [4].

2 Improper 2-Choosability

2.1 Lower Bound for $M(k,2)$

In this subsection, we shall prove Theorem 1. Note that if $k = 0$ then Theorem 1 holds trivially. Indeed a graph with maximum average degree less than 2 contains no cycle and so it is a forest. Hence it is 2-choosable. Furthermore $M(0,2) \leq 2$ since an odd cycle is not 2-colourable, so $M(0,2) = 2$. For bigger values of k, we will need the following preliminary definitions and results:

Definition 1. If $v \in V(G)$ then $d_G(v)$ denotes the degree of v in the graph G. For all positive integer p, a vertex of degree equal to (resp. at most, resp. at least) p is called a *p-vertex* (resp. *$(\leq p)$-vertex*, resp. *$(\geq p)$-vertex*). For $S \subseteq V(G)$ (resp. $E \subseteq E(G)$) we denote by $G - S$ (resp. $G - E$) the induced subgraph of G obtained by removing the vertices (resp. edges) of S (resp. E) from $V(G)$ (resp. $E(G)$). If $S = \{v\}$ and $E = \{uv\}$, we shall note $G - v = G - S$ and $G - uv = G - E$. The *union* (resp. *intersection*) of the graphs G_1 and G_2 is the graph $G = G_1 \cup G_2$ (resp. $G = G_1 \cap G_2$) such that $V(G) = V(G_1) \cup V(G_2)$ (resp. $V(G) = V(G_1) \cap V(G_2)$) and $E(G) = E(G_1) \cup E(G_2)$ (resp. $E(G) = E(G_1) \cap E(G_2)$). Let D be a digraph and u one of its vertices. An *outneighbour* (resp. *inneighbour*) of u in D is a vertex v of D such that there exists an arc from u to v (resp. from v to u) in D. The *outdegree* (resp. *indegree*) of u in D, denoted by $d_D^+(u)$ (resp. $d_D^-(u)$), is the number of outneighbours (resp. inneighbours) of u in D. The *degree* of u is $d_D(u) = d_D^-(u) + d_D^+(u)$; it is the degree of u in the underlying undirected graph.

A graph is said to be $(k, 2)$-*minimal* if it is not k-improper 2-choosable but each of its proper subgraphs is.

The idea of the proof of Theorem 1 is to consider a $(k, 2)$-minimal graph and apply a discharging procedure, the rule of which is to discharge $\frac{k}{k+2}$ along the arcs of a discharging digraph which is obtained using the following process:

1. Orient each edge uv where v is a 2-vertex from u to v.
2. If $k \geq 3$, orient each edge uv where v is a 3-vertex from u to v.
3. While there is an unoriented edge uv where v an i-vertex with outdegree $i - 1$ for some $k + 2 \leq i < \frac{3k}{2} + 2$, we orient it from u to v.

The digraph D induced by the oriented edges is called a *discharging digraph* of G.

The aim of the next lemmata is to establish some properties of such a discharging digraph.

Lemma 1 (Škrekovski [7]). *Let $k \geq 1$ and let G be a $(k, 2)$-minimal graph. Then G has minimum degree at least 2 and two $(\leq k + 1)$-vertices are not adjacent.*

Definition 2. If u and v are two vertices of a digraph D, a (u, v)-dipath is a directed path from u to v. The *outsection* of u in D, denoted $A_D^+(u)$, is the set of vertices v such that there is a (u, v)-dipath in D.

An *arborescence* is an oriented tree in which every path is directed from a vertex called the *root*. Note that in an arborescence every vertex except the root has indegree 1. The *leaves* of the arborescence are the vertices of outdegree 0. A vertex which is neither a leaf nor the root is an *internal vertex*. A *quasi-arborescence* is a directed graph obtained from an arborescence by identifying some leaves.

Lemma 2. *Let D be a discharging digraph of a $(k, 2)$-minimal graph, and $k \geq 1$.*

- *D has no 2-circuit since by Lemma 1 two $(\leq k + 1)$-vertices cannot be adjacent. So it has no circuit at all.*
- *If $k \leq 2$, only vertices of degree 2 or $k + 2$ have indegree more than zero.*
- *Every 2-vertex has indegree 2 in D and if $k \geq 3$, every 3-vertex has indegree 3.*
- *For every vertex u, $A_D^+(u)$ is a quasi-arborescence whose leaves have degree 2 (resp. 2 or 3) in G if $k \leq 2$ (resp. $k \geq 3$). In particular, the indegree of the leaves in $A_D^+(u)$ is at most 2 (resp. 3).*

Definition 3. A quasi-arborescence is a $(k, 2)$-*quasi-arborescence* if and only if every vertex has outdegree at most $\max\{2, 2k - 1\}$ and every leaf has indegree at most $\min\{k, 3\}$.

Lemma 3. *Let $k \geq 2$. Let Q be a $(k,2)$-quasi-arborescence rooted at u and L a 2-list-assignment of Q. Then any L-colouring of the leaves can be extended to a k-improper L-colouring of D such that u has impropriety at most $k-1$.*

Proof. By induction on the number of vertices of Q, the result being trivially true if $|V(Q)| = 1$.

Suppose now that $|V(Q)| > 1$ and the result holds for smaller k-quasi-arborescences. Let $v_1, \ldots, v_s$ be the outneighbours of u in Q. Note that $Q - u$ is the union of s $(k,2)$-quasi-arborescences Q_i, $1 \leq i \leq s$ rooted at v_i that are disjoint except possibly on their leaves.

Let c be an L-colouring of the leaves of Q. Then by induction it can be extended to a k-improper L-colouring of each of the Q_i so that $im(v_i) \leq k-1$. Since a leaf of Q has indegree at most $\min\{k,3\}$ and $\mathrm{im}_Q(x) = \mathrm{im}_{Q_i}(x)$ for every vertex of Q_i which is not a leaf, then the union of these colourings is a k-improper L-colouring of Q such that $im(v_i) \leq k-1, 1 \leq i \leq s$.

Now, one of the two colours of $L(u)$, say α, is assigned to at most $k-1$ neighbours of u since $s \leq 2k-1$. Thus setting $c(u) = \alpha$, we obtain the desired colouring.

Obviously, the above result cannot be extended for $k = 1$ because it is hopeless to extend every L-colouring of the leaves in a colouring such that the root has impropriety 0. However, the following weaker result holds:

Lemma 4. *Let Q be a $(1,2)$-quasi-arborescence rooted at u, L a 2-list-assignment of Q with $L(u) = \{\alpha, \beta\}$ and c an L-colouring of S, the set of leaves of Q with indegree 1. One of the following holds:*

(i) c can be extended to a 1-improper L-colouring of Q such that $im(u) = 0$;
(ii) c can be extended to two different 1-improper L-colourings of Q c_1 and c_2 such that $c_1(v) = c_2(v)$ if $v \neq u$.

Lemma 5. *Let $k \geq 3$. Let D be a discharging digraph of a $(k,2)$-minimal graph G.*

(i) Every i-vertex with $4 \leq i \leq k+1$ has outdegree zero.
(ii) Every i-vertex with $k+2 \leq i \leq 2k+1$ has outdegree less than i.

Proof. (i) Suppose, for a contradiction, that v is a vertex contradicting the assertion and let u be an outneighbour of v. Note that u is a $(< \frac{3k}{2} + 2)$-vertex by definition of a discharging digraph.

Let L be a 2-list-assignment of G. Let S be the set of leaves of $A^+_D(u)$. By minimality, let c be a k-improper L-colouring of $G - A^+_D(u)$.

$A^+_D(u)$ is a $(k,2)$-quasi-arborescence: since u is dominated by v in D, u has outdegree less than $\frac{3k}{2} + 1$, and so at most $2k-1$. Thus, by Lemma 3, we can extend c to $G - vu$ so that $im(u) \leq k-1$. Since the leaves have degree at most $3 \leq k$, the impropriety of the leaves is at most $3 \leq k$. So we obtain a k-improper L-colouring of $G - uv$.

If $c(u) \neq c(v)$ or $\mathrm{im}_{G-uv}(v) \leq k-1$ then c is a k-improper L-colouring of G. Otherwise all the $k+1$ neighbours of v are coloured by the same colour so recolouring v with its other allowed colour yields a k-improper L-colouring of G.

Hence G is k-improper 2-choosable which is a contradiction.

(ii) Suppose, for a contradiction, that v is an i-vertex contradicting the assertion.

Let L be 2-list-assignment of G and c a k-improper L-colouring of $G - v$. There is a colour of $L(v)$, say α, that is assigned to at most k neighbours of v. Let $v_1, \ldots, v_s$ be these neighbours.

Let $G' = G - \bigcup_{j=1}^{s} A_D^+(v_j)$. And set $c' = c$ for every vertex of G' and every leaf of the $A_D^+(v_j)$. By Lemma 3 applied to each $A_D^+(v_j)$ (which are disjoint except possibly on their leaves), we can extend c' into a k-improper L-colouring of $G - v$ so that $im(v_j) \leq k-1$ for $1 \leq j \leq s$. Now by definition of c', the only neighbours of v that may be assigned α by c' are those of $\{v_1, \ldots, v_s\}$. Hence setting $c'(v) = \alpha$, the L-colouring c' is k-improper.

Hence G is k-improper 2-choosable which is a contradiction.

Analogously, one can prove the following two lemmata:

Lemma 6. *Let D be a discharging digraph of a $(2,2)$-minimal graph G.*

(i) The outdegree of a 3-vertex is zero.
(ii) If v is an i-vertex with $i \in \{4,5\}$ then its outdegree is less than i.

Lemma 7. *Let D be a discharging digraph of a $(1,2)$-minimal graph G. There is no 3-vertex with outdegree 3 in D.*

Proof (of Theorem 1). Let G be a $(k,2)$-minimal graph and D a discharging digraph of G. We start with a charge $w(v) = d(v)$ on each vertex and we apply the following discharging rule: every vertex gives $\frac{k}{k+2}$ to each of its outneighbours.

Let us examine the new charge $w'(v)$ of a vertex v, regarding its degree:

- If v is a 2-vertex then it has indegree 2 so its new charge is $w'(v) = 2 + \frac{2k}{k+2} = \frac{4k+4}{k+2}$.
- If v is a 3-vertex and $k \geq 3$, then it has indegree 3 so its new charge is $w'(v) = 3 + 3 \times \frac{k}{k+2} = \frac{6k+6}{k+2} > \frac{4k+4}{k+2}$. If v is a 3-vertex and $k = 2$ then it has outdegree 0 by Lemma 6 and indegree 0 by the construction and hence $w'(v) = 3$.
- If $4 \leq d(v) \leq k+1$, $(k \geq 3)$, then by Lemma 5 (i), v has outdegree 0 so its charge is $d(v) \geq 4 > \frac{4k+4}{k+2}$.
- If $k+2 \leq d(v) < \frac{3k}{2}+2$ then either v has outdegree at most $d(v)-2$ and so its new charge is at least $d(v) - (d(v)-2) \times \frac{k}{k+2} = \frac{2d(v)}{k+2} + \frac{2k}{k+2} \geq 2 + \frac{2k}{k+2} = \frac{4k+4}{k+2}$, or by Lemmata 5-7, it has outdegree $d(v) - 1$. In this case, by definition of a discharging digraph, v has indegree 1 so its new charge is:
$d(v) - (d(v)-1) \times \frac{k}{k+2} + \frac{k}{k+2} = d(v) - (d(v)-2) \times \frac{k}{k+2} \geq \frac{4k+4}{k+2}$.
- If $\frac{3k}{2}+2 \leq d(v) \leq 2k+1$, $(k \geq 2)$, then by Lemmata 5 and 6, v has outdegree at most $d(v)-1$. So $w'(v) \geq d(v) - (d(v)-1) \times \frac{k}{k+2} = \frac{2d(v)}{k+2} + \frac{k}{k+2} \geq \frac{3k+4+k}{k+2} = \frac{4k+4}{k+2}$.

– If $d(v) \geq 2k+2$, then $w'(v) \geq d(v)(1 - \frac{k}{k+2}) = \frac{2d(v)}{k+2} \geq \frac{4k+4}{k+2}$.

Hence $\text{Mad}(G) \geq \frac{1}{|V|} \sum_{v \in V} d(v) = \frac{1}{|V|} \sum_{v \in V} w'(v) \geq \frac{4k+4}{k+2}$.

2.2 Upper Bound for $M(k, 2)$

Let us fix $k \geq 1$. In this subsection, we shall construct a family of graphs $(G_n^k)_{n \geq 1}$ such that for all $n \geq 1$:

- G_n^k is not k-improper 2-colourable.
- $\text{Mad}(G_n^k) = \dfrac{2n(4k^2+6k+4)+4k^2+6k+2}{2n(k^2+2k+2)+(k+1)^2}$.

Hence we will deduce Theorem 2. We denote by H_k the graph composed of two adjacent vertices u and v also connected by $k+1$ internally disjoint paths of length 2. Take k copies of H_k and create the graph F_k by identifying the vertices v of each copy. Note that F_k has one vertex of degree $k(k+2)$, k vertices of degree $k+2$ and $k(k+1)$ vertices of degree 2. Now we take $2n+1$ copies of F_k and we join the vertices v of each copy creating a cycle of size $2n+1$. At last we make a subdivision of all the edges of the cycle but one so as to obtain the graph G_n^k.

Lemma 8. *G_n^k is not k-improper 2-colourable.*

As it is easily seen, the maximum average degree of G is its average degree, which is equal to M_n^k.

3 Improper l-Choosability, $l \geq 2$

3.1 Lower Bound for $M(k, l)$

In this subsection, we shall prove Theorem 3. The result of the theorem is trivial if $k = 0$ since a graph of maximum average degree less than l is $(l-1)$-*degenerate* (i.e. each of its subgraphs has a vertex of degree at most $l-1$). Hence it is l-choosable. For bigger values of k, we will need some preliminary results.

Definition 4. A graph is said to be (k, l)-*minimal* if it is not k-improper l-choosable but each of its proper subgraphs is.

Lemma 9. *Let G be a graph, L a list-assignment and c an L-colouring. If a vertex v has impropriety at least $d(v) - |L(v)| + 2$ under c, then there exists an L-colouring c' of G such that $c'(u) = c(u)$ if $u \neq v$ and $\text{im}_{c'}(v) = 0$.*

We now generalise Lemmata 1, 3 and 4.

Lemma 10. *Let $k \geq 1$ and let G be a (k, l)-minimal graph. Then G has minimum degree at least l and two $(\leq l+k-1)$-vertices are not adjacent.*

Definition 5. Let G be a (k,l)-minimal graph. We partially orient G using the following process:

1. Orient each edge uv where v is a $(\leq l+k-1)$-vertex from u to v.
2. While there is an i-vertex v with outdegree exactly $i-l+1$ and indegree 0 for some $l+k \leq i < l+k+\frac{k}{l}$, we orient one of its unoriented incident edges uv from u to v.

The digraph D induced by the oriented edges is called a *discharging digraph* of G. Note that only vertices of degree less than $l+k+\frac{k}{l}$ can have indegree more than zero, and for $i \leq l+k-1$, every i-vertex has indegree exactly i in D.

A quasi-arborescence rooted at u is a (k,l)*-quasi-arborescence* if and only if every vertex has outdegree at most $\max\{2, 2k-1\}$ and every leaf has indegree at most $l+k-1$.

Lemma 11. *Let $k \geq 2$ and let Q be a (k,l)-quasi-arborescence rooted at u. Let L be a list-assignment of Q such that $|L(v)| \geq \max\{1, d_Q(v)-k+1\}$ if v is a leaf and $|L(v)| \geq 2$ otherwise. We denote by S the set of leaves that have indegree at least $k+1$ in Q (and hence a colour-list of size at least 2). Any L-colouring of the leaves extends in an L-colouring of Q such that:*

- $im(u) \leq k-1$.
- $\forall v \notin S, im(v) \leq k$.

Furthermore, possibly by recolouring some vertices of S, this L-colouring of G can be made k-improper.

The above result cannot be extended for $k=1$. However the following result holds:

Lemma 12. *Let Q be a $(1,l)$-quasi-arborescence rooted at u and L any list-assignment of Q such that $|L(v)| \geq 2$ if v is not a leaf, and $|L(v)| \geq d_Q(v)$ otherwise. We denote by S the set of leaves with indegree at least 2. Let c be an L-colouring of the leaves. One of the followings holds:*

(i) *c can be extended to an L-colouring of Q such that $im(u)=0$ and $im(v) \leq 1$ if $v \notin S$;*
(ii) *c can be extended to two different L-colourings of Q c_1 and c_2 such that $c_1(v)=c_2(v)$ if $v \neq u$ and $im^{c_i}(v) \leq 1$ if $v \notin S$.*

Furthermore, possibly by recolouring vertices of S, all these L-colourings can be made 1-improper.
Moreover, if $|L(u)| \geq 3$ then (i) holds.

Using these results, we can say more about the structure of a discharging digraph. The following lemma generalises Lemma 2.

Lemma 13. *Let D be a discharging digraph of a (k,l)-minimal graph G.*

(i) *Every vertex u with $l+k \leq d(u) \leq l+2k-1$ has outdegree at most $d(u)-l+1$. In particular, D is acyclic.*

(ii) For every vertex u with indegree 1, $A_D^+(u)$ is a (k,l)-quasi-arborescence. In particular, the indegree of the leaves in $A_D^+(u)$ is at most $l+k-1$.

Proof (of Theorem 3). Let G be a (k,l)-minimal graph and D a discharging digraph of G. We start with a charge $w(v) = d(v)$ on each vertex and we apply the following discharging rule: every vertex gives $\frac{k}{l+k}$ to each of its outneighbours. One can check that, by using Lemma 13, the new charge of every vertex is at least $l + \frac{lk}{l+k}$.

3.2 Upper Bound for $M(k,l)$

In this subsection we shall construct for all $l \geq 2$ and all $k \geq 1$, a graph G_l^k which is not k-improper l-colourable. So its maximum average degree will give an upper bound for $M(k,l)$. To construct G_2^k, take $k+1$ copies of H_k (defined in Subsection 2.2) and identify their vertex v. We define G_l^k, $l \geq 3$, inductively. First we create the graph M_l^k by taking k copies of G_{l-1}^k and adding a vertex w which we join to every other vertices. Then we take $l-1$ copies $M^1, \ldots, M^{l-1}$ of M_l^k and we join all the vertices $w_1, \ldots, w_{l-1}$ (so that they form a complete graph of size $l-1$). Now, we add $k+2$ vertices $z_0, z_1, \ldots, z_{k+1}$ each joined to each of the w_i, $1 \leq i \leq l-1$. Last we add the edges $z_0 z_i$ for $1 \leq i \leq k+1$.

Lemma 14. *For all $l \geq 2$ and all $k \geq 1$, the graph G_l^k is not k-improper l-colourable.*

Proposition 2. $\mathrm{Mad}(G_l^k)$ *tends to $2l$ as k tends to infinity.*

Proof. It is clear that the maximum average degree of G_l^k is its average degree.

The number of vertices of G_l^k is $n_l^k = 2l + (l+1)k + \displaystyle\sum_{i=2}^{l} \frac{(l-1)!}{(l-i)!} k^i$. Indeed n_l^k satisfies: $n_2^k = k^2 + 3k + 3$ and $\forall l \geq 3, n_l^k = (k \times n_{l-1}^k + 1) \times (l-1) + k + 2$. In particular, as a polynomial in k, $n_l^k \sim (l-1)! k^l$.

Let s_l^k denotes the sum of the degrees of the vertices in G_l^k. s_l^k satisfies: $s_2^k = 4k^2 + 10k + 6$ and $s_l^k = (l-1)(k \times s_{l-1}^k + 2k \times n_{l-1}^k + l + k) + (l+1)k + 2l$ if $l \geq 3$. Hence it is a polynomial in k of degree l. Furthermore, denoting by c_l^k its dominant coefficient, we have: $c_2^k = 4$ and $\forall l \geq 3, c_l^k = (l-1) \times c_{l-1}^k + 2k \times (l-1)!$. Thus $c_l^k = 2l!$. So $s_l^k \sim 2l! k^l$.

Hence the limit of $\mathrm{Mad}(G_l^k)$ as k tends to infinity is $2\frac{l!}{(l-1)!} = 2l$.

Corollary 1 immediately follows from Theorem 3 and Proposition 2.

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