Improper choosability of graphs and maximum average degree

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Abstract

Improper choosability of planar graphs has been widely studied. In particular, Škrekovski investigated the smallest integer g_k such that every planar graph of girth at least g_k is k-improper 2-choosable. He proved [9] that $6 \leq g_1 \leq 9$; $5 \leq g_2 \leq 7$; $5 \leq g_3 \leq 6$ and $\forall k \geq 4, g_k = 5$. In this paper, we study the greatest real $M(k,l)$ such that every graph of maximum average degree less than $M(k,l)$ is k-improper l-choosable. We prove that if $l \geq 2$ then $M(k,l) \geq l + \frac{lk}{l+k}$. As a corollary, we deduce that $g_1 \leq 8$ and $g_2 \leq 6$, and we obtain new results for graphs of higher genus. We also provide an upper bound for $M(k,l)$. This implies that for any fixed l, $M(k,l) \xrightarrow[k\to\infty]{} 2l$.

keywords: improper colouring, choosability, maximum average degree, planar graph, girth, genus.

1 Introduction

Let G be a graph. We note $V(G)$ its vertex set and $E(G)$ its edge set.

A *colouring* is a function from the vertex set into a set of colours S. If $|S| = l$ we call it l-colouring. Let c be a colouring of G. The *impropriety* of a vertex v in G under c, denoted by $im_G^c(v)$, is the number of neighbours u of v in G such that $c(u) = c(v)$. The *impropriety* of c in G is $im_G(c) = \max\{im_G^c(v) — \ v \in V(G)\}$. A colouring is *$k$-improper* if its impropriety is at most k and a graph is *k-improper l-colourable* if it admits a k-improper l-colouring. The *k-improper chromatic number* of G, denoted by $c_k(G)$, is the smallest integer l such that G is k-improper l-colourable. Note that 0-improper colouring is the usual notion of proper colouring, so the 0-improper chromatic number is exactly the chromatic number usually denoted $\chi(G)$.

One can analogously generalise the notion of *choosability*. A *list-assignment* of a graph G is a function L which assigns to each vertex $v \in V(G)$ a prescribed list of colours $L(v)$. L is an l-list-assignment provided each list is of size at least l. G is k-improper L-colourable if there exists a k-improper colouring c of G such that $\forall v \in V(G), v \in L(v)$. In this case, c is a k-improper L-colouring of G. G is k-improper l-choosable if it is k-improper L-colourable for every l-list-assignment L.

Colourings of planar graphs have been widely studied. In particular p_k and p_k^*, the smallest integers l such that every planar graph is k-improper l-colourable and k-improper l-choosable respectively, are known for almost all k. Indeed Thomassen showed in [10] that every planar graph is 5-choosable and there are planar graphs which are not 4-choosable [13] so $p_0^* = 5$. Every planar graph is 4-colourable [1, 2] and there are planar graphs which are not 1-improper 3-colourable, so $p_0 = p_1 = 4$. But we do not know the exact value of p_1^* which is either 4 or 5. However, it is conjectured that it is 4.

Conjecture 1 (Eaton and Hull [3], Škrekovski [7]) *Every planar graph is 1-improper 4-choosable.*

As shown independently by Eaton and Hull [3] and Škrekovski [7], every planar graph is 2-improper 3-choosable and for every k, there are planar graphs which are not k-improper 2-colourable. Hence $p_k = p_k^* = 3$ for any $k \geq 2$.

Moreover improper colourings of planar graphs have also been studied under some girth restrictions. The *girth* of a graph is the smallest length of a cycle. The well-known theorem of Grötzsch [4, 12] states that every planar graph of girth at least 4 is 3-colourable. Voigt [14] showed a planar graph of girth 4 which is not 3-choosable and Thomassen [11] proved that every planar graph of girth at least 5 is 3-choosable. In [8], Škrekovski showed that every planar graph of girth at least 4 is 1-improper 3-choosable. In [9], Škrekovski investigated k-improper 2-choosability of planar graphs in relation with their girth. Denoting by g_k the smallest integer such that every planar graph of girth at least g_k is k-improper 2-choosable, he proved $6 \leq g_1 \leq 9$, $5 \leq g_2 \leq 7$, $5 \leq g_3 \leq 6$ and $\forall k \geq 4, g_k = 5$. Hence the only unknown values are g_1, g_2 and g_3.

In this paper, we study the k-improper l-choosability of graphs in relation with their maximum average degree. The *average degree* of a graph G, denoted by $\mathrm{Ad}(G)$, is the sum of the degree of each vertex divided by the number of vertices. The *maximum average degree* of G, denoted by $\mathrm{Mad}(G)$, is the maximum of the average degree of each of its subgraphs (including G). If G is not a forest, the *heart* of G, denoted by $h(G)$, is the biggest subgraph of G in which every vertex has degree at least 2. It can be obtained by consecutive removing of vertices of degree 1.

Proposition 1 *If G is not a forest, then* $\mathrm{Mad}(G) = \mathrm{Mad}(h(G))$.

Proof. As $h(G)$ is a subgraph of G, $\mathrm{Mad}(G) \geq \mathrm{Mad}(h(G))$. Let H be a subgraph of G such that $\mathrm{Mad}(G) = \mathrm{Ad}(H)$. Then H is not a forest since otherwise we would have $\mathrm{Mad}(G) < 2$ and G would be a forest. So $h(H)$ is defined and it is a subgraph of $h(G)$. Moreover, $h(H)$ has minimum degree at least 2, so adding to it vertices of degree 1 cannot increase its average degree: let H' be a supergraph obtained from $h(H)$ by adding $k \geq 1$ vertices of degree 1. We assume that $h(H)$ has n vertices. Then

$$\mathrm{Ad}(H') = \frac{n \times \mathrm{Ad}(h(H)) + 2k}{n+k} = \mathrm{Ad}(h(H)) + \frac{2k - k \times \mathrm{Ad}(h(H))}{n+k} \leq \mathrm{Ad}(h(H))$$

since $\mathrm{Ad}(h(H)) \geq 2$. So $\mathrm{Mad}(h(G)) \geq \mathrm{Ad}(h(H)) \geq \mathrm{Ad}(H) = \mathrm{Mad}(G)$. □

Let $M(k,l)$ be the greatest real such that every graph of maximum average degree less than $M(k,l)$ is k-improper l-choosable. Obviously, $M(k_1,l) \leq M(k_2,l)$ if $k_1 \leq k_2$. We have $M(k,1) = \frac{2k+2}{k+2}$ since a graph is k-improper 1-choosable if and only if it has maximum degree at most k (and a graph of maximum degree at least $k+1$ contains the star S_{k+1} as a subgraph, so its maximum average degree is at least $\frac{2k+2}{k+2}$). If $l \geq 2$, first note that any tree is 0-improper 2-choosable. Furthermore, for any $k \geq 0$, a graph G which is not a forest is k-improper 2-choosable if and only if its heart is. Hence, we shall restrict the study to graphs with minimum degree at least 2.

In order to introduce our method which uses some discharging process, we first present it in Section 2 for improper 2-choosability, where we prove the following results.

Theorem 1 *For every $k \geq 0$, all graphs of maximum average degree less than $\frac{4k+4}{k+2}$ are k-improper 2-choosable.*

Theorem 2 *For all $k \geq 1$, $M(k,2) \leq \dfrac{4k^2+6k+4}{k^2+2k+2} = 4 - \dfrac{2k+4}{k^2+2k+2}$.*

In Section 3 we extend the lower bound of Section 2 to any value of l.

Theorem 3 *For every $l \geq 2$ and every $k \geq 0$, all graphs of maximum average degree less than $\frac{l(l+2k)}{l+k}$ are k-improper l-choosable.*

We also provide, for any value of l and k, a graph which is not k-improper l-choosable, and we get the following.

Corollary 1 *For any fixed l, $\lim_{k \to +\infty} M(k,l) = 2l$.*

The girth and the maximum average degree of a planar graph are related to each other.

Theorem 4 *If G is a planar graph of girth g then*

$$\mathrm{Mad}(G) < 2 + \frac{4}{g-2}.$$

Proof. The result is true if G is a tree since it is 1-degenerate. So we can assume that g is finite. We recall Euler's formula for a planar graph H: $|V(H)| - |E(H)| + |F(H)| = 2$ with $|F(H)|$ the number of faces of H. Note that every subgraph H of G has girth at least g, so $g|F(H)| \le 2|E(H)|$. Thus $2g - g|V(H)| + g|E(H)| = g|F(H)| \le 2|E(H)|$. Hence $\frac{2|E(H)|}{|V(H)|} \le \frac{2g}{g-2} - \frac{4g}{(g-2)|V(H)|} < \frac{2g}{g-2}$ for every subgraph H of G. □

So as a corollary of Theorem 1, we obtain the following upper bounds for g_k which are better than Škrekovski's ones.

Corollary 2 *Let G be a planar graph of girth g.*

1. *If $g \ge 8$ then G is 1-improper 2-choosable, so $g_1 \le 8$.*
2. *If $g \ge 6$ then G is 2-improper 2-choosable, so $g_3 \le g_2 \le 6$.*
3. *If $g \ge 5$ then G is 4-improper 2-choosable, so $g_k \le 5$ for $k \ge 4$.*

At last, we show in Subsection 3.3 how Theorem 3 implies results for graphs of higher genus, improving for instance the previously known results of Miao [5].

2 Improper 2-choosability

2.1 Lower bound for $M(k, 2)$

In this subsection, we shall prove Theorem 1.

Note that if $k = 0$ Theorem 1 holds trivially. Indeed a graph with maximum average degree less than 2 contains no cycle and so is a forest. Hence it is 2-choosable. Furthermore $M(0, 2) \le 2$ since an odd cycle is not 2-colourable, so $M(0, 2) = 2$.

For bigger values of k, we will need the following preliminary definitions and results.

Definition 1 If $v \in V(G)$ then $d_G(v)$ denotes the degree of v in the graph G. For any positive integer d, a vertex of degree equals to (resp. at most, resp. at least) d is called a d-vertex (resp. $(\le d)$-vertex, resp. $(\ge d)$-vertex). For $S \subseteq V(G)$ (resp. $E \subseteq E(G)$) we denote by $G - S$ (resp. $G - E$) the induced subgraph of G obtained by removing the vertices (resp. edges) of S (resp. E) from $V(G)$ (resp. $E(G)$). If $S = \{v\}$ and $E = \{uv\}$, we shall note $G - v = G - S$ and $G - uv = G - E$. The union (resp. intersection) of the graphs G_1 and G_2 is the graph $G = G_1 \cup G_2$ (resp. $G = G_1 \cap G_2$) such that

$V(G) = V(G_1) \cup V(G_2)$ (resp. $V(G) = V(G_1) \cap V(G_2)$) and $E(G) = E(G_1) \cup E(G_2)$ (resp. $E(G) = E(G_1) \cap E(G_2)$).

A graph is said to be $(k,2)$-minimal if it is not k-improper 2-choosable but each of its proper subgraphs is.

Lemma 1 (Škrekovski [9]) *Let $k \geq 1$ and let G be a $(k,2)$-minimal graph.*

(i) *The minimum degree of G is at least 2.*

(ii) *Two $(\leq k+1)$-vertices are not adjacent.*

Definition 2 Let D be a digraph. The outdegree (resp. indegree) of a vertex u in D is denoted by $d_D^+(u)$ (resp. $d_D^-(u)$). The *degree* of u is $d_D(u) = d_D^-(u) + d_D^+(u)$; it is the degree of u in the underlying undirected graph.

If u and v are two of its vertices, a (u,v)-dipath is a directed path from u to v. The *outsection* of u in D, denoted $A_D^+(u)$, is the set of vertices v such that there is a (u,v)-dipath in D.

An *arborescence* is an oriented tree in which every path is directed from a vertex called the *root*. Note that in an arborescence every vertex except the root has indegree 1. The *leaves* of the arborescence are the vertices of outdegree 0. A vertex which is neither a leaf nor the root is an *internal vertex*. A *quasi-arborescence* is a directed graph obtained from an arborescence by identifying some leaves.

Let G be a $(k,2)$-minimal graph. We partially orient G using the following process.

1. Orient each edge uv where v is a 2-vertex from u to v.
2. If $k \geq 3$, orient each edge uv where v is a 3-vertex from u to v.
3. While there is an unoriented edge uv where v an i-vertex with $k+2 \leq i < \frac{3k}{2}+2$ and outdegree $i-1$, we orient it from u to v.

The digraph D induced by the oriented edges is called a *discharging digraph* of G.

The following proposition, whose proof is left to the reader, follows immediately from the definition of a discharging digraph.

Proposition 2 *Let D be a discharging digraph of a $(k,2)$-minimal graph.*

- *The digraph D has no 2-circuit since two $(\leq k+1)$-vertices are not adjacent by Lemma 1* (ii). *So it has no circuit at all.*
- *If $k \leq 2$, only vertices of degree 2 or $k+2$ have indegree more than zero.*
- *Every 2-vertex has indegree 2 in D and if $k \geq 3$, every 3-vertex has indegree 3.*
- *For every vertex u, $A_D^+(u)$ is a quasi-arborescence whose leaves have degree 2 (resp. 2 or 3) in G if $k \leq 2$ (resp. $k \geq 3$). In particular, the indegree of the leaves in $A_D^+(u)$ is at most 2 (resp. 3).*

Definition 3 A quasi-arborescence is a $(k,2)$-quasi-arborescence if and only if:

- every vertex has outdegree at most $\max\{2, 2k-1\}$; and
- every leaf has indegree at most $\min\{k, 3\}$.

Lemma 2 *Let $k \geq 2$. Let Q be a $(k,2)$-quasi-arborescence rooted at u and L a 2-list-assignment of Q. Then any L-colouring of the leaves can be extended in a k-improper L-colouring of D such that u has impropriety at most $k-1$.*

Proof. By induction on the number of vertices of Q, the result being trivially true if $|V(Q)| = 1$.

Suppose now that $|V(Q)| > 1$ and the result holds for smaller k-quasi-arborescences. Let $v_1, \dots, v_s$ be the outneighbours of u in Q. Note that $Q - u$ is the union of s $(k,2)$-quasi-arborescences Q_i, $1 \leq i \leq s$ rooted at v_i that are disjoint except possibly on their leaves.

Let c be an L-colouring of the leaves of Q. Then by induction it can be extended in a k-improper L-colouring of each of the Q_i so that $im(v_i) \leq k-1$. Since a leaf of Q has indegree at most $\min\{k, 3\}$ and $im_Q(x) = im_{Q_i}(x)$ for every vertex of Q_i which is not a leaf, then the union of these colourings is a k-improper L-colouring of Q such that $im(v_i) \leq k-1, 1 \leq i \leq s$.

Now, one of the two colours of $L(u)$, say α, is assigned to at most $k-1$ neighbours of u since $s \leq 2k-1$. Thus setting $c(u) = \alpha$, we obtain the desired colouring. □

Obviously, the above result cannot be extended for $k = 1$ because it is hopeless to extend every L-colouring of the leaves in a colouring such that the root has impropriety 0. However, one can prove the following weaker result.

Lemma 3 *Let Q be a $(1,2)$-quasi-arborescence rooted at u, L a 2-list-assignment of Q with $L(u) = \{\alpha, \beta\}$ and c an L-colouring of S, the set of leaves of Q with indegree 1. One the following holds.*

(i) *The colouring c can be extended in a 1-improper L-colouring of Q such that $im(u) = 0$.*

(ii) *The colouring c can be extended in two different 1-improper L-colourings of Q c_1 and c_2 such that $c_1(v) = c_2(v)$ if $v \neq u$.*

Proof. We proceed by induction on the number of vertices of Q. Let v_1 an v_2 be two outneighbours of u in Q. $Q - u$ is the union of two $(1,2)$-quasi-arborescences Q_1 and Q_2, rooted at v_1 and v_2 respectively, that are disjoint except possibly on their leaves. Let S' be the set of leaves in $Q_1 \cap Q_2$ and $L(u) = \{\alpha, \beta\}$. We L-colour the leaves of Q_i that have indegree 1 in Q_i. By induction, each of the Q_i satisfies (i) or (ii).

If at least one of the Q_i satisfies (ii), then one can extend c to $Q_1 \cup Q_2$ such that $\{c(v_1), c(v_2)\} \neq L(u)$, say $\alpha \notin \{c(v_1), c(v_2)\}$. Moreover for any vertex x not in $V(Q_i) \setminus S'$,

$im_Q(x) = im_{Q_i}(x) \leq 1$. If a vertex $s' \in S'$ has impropriety 2 then its two neighbours are coloured the same. So recolouring s' with the colour of $L(s')\setminus\{c(s')\}$, we get a 1-improper L-colouring of $Q_1 \cup Q_2$. Hence setting $c(u) = \alpha$, we get a 1-improper L-colouring of Q such that $im(u) = 0$. Thus Q satisfies (i).

Suppose now Q_1 and Q_2 both satisfy (i). Then, possibly with recolouring of vertices of S' as before, one can extend c into a 1-improper L-colouring of $Q_1 \cup Q_2$ such that $im(v_1) = im(v_2) = 0$. If $\{c(v_1), c(v_2)\} \neq L(u)$, say $\alpha \notin \{c(v_1), c(v_2)\}$ then setting $c(v) = \alpha$, we get a 1-improper L-colouring of Q such that $im(u) = 0$. Thus Q satisfies (i). If not then assigning to u the colours α and β, we get the two 1-improper L-colourings of Q satisfying (ii). □

Lemma 4 *Let $k \geq 3$. Let D be a discharging digraph of a $(k, 2)$-minimal graph G.*

(i) *Every i-vertex with $4 \leq i \leq k + 1$ has outdegree zero.*

(ii) *Every i-vertex with $k + 2 \leq i \leq 2k + 1$ has outdegree less than i.*

Proof.

(i) Suppose, for a contradiction, that v is a vertex contradicting the assertion and let u be an outneighbour of v. Note that u is a $(< \frac{3k}{2} + 2)$-vertex by definition of a discharging digraph.

Let L be a 2-list-assignment of G. Let S be the set of leaves of $A^+_D(u)$. By minimality, let c be a k-improper L-colouring of $G - A^+_D(u)$.

$A^+_D(u)$ is a $(k, 2)$-quasi-arborescence: since it is dominated by v in D, u has outdegree less than $\frac{3k}{2} + 1$ and so at most $2k - 1$. Thus, by Lemma 2, we can extend c to $G - vu$ so that $im(u) \leq k - 1$. Since the leaves have degree at most $3 \leq k$, the impropriety of the leaves is at most $3 \leq k$. So we obtain a k-improper L-colouring of $G - uv$.

If $c(u) \neq c(v)$ or $im_{G-uv}(v) \leq k - 1$ then c is a k-improper L-colouring of G. Otherwise all the $k + 1$ neighbours of v are coloured the same so recolouring v with its other allowed colour yields a k-improper L-colouring of G.

Hence G is k-improper 2-choosable which is a contradiction.

(ii) Suppose, for a contradiction, that v is an i-vertex contradicting the assertion.

Let L be 2-list-assignment of G and c a k-improper L-colouring of $G - v$. There is a colour of $L(v)$, say α, that is assigned to at most k neighbours of v. Let $v_1, \ldots, v_s$ be these neighbours.

Let $G' = G - \bigcup_{j=1}^{s} A^+_D(v_j)$. And set $c' = c$ for every vertex of G' and every leaf of the $A^+_D(v_j)$. By Lemma 2 applied to each $A^+_D(v_j)$ (which are disjoint except possibly on their leaves), we can extend c' into a k-improper L-colouring of $G - v$ such that $im(v_j) \leq k - 1$ for $1 \leq j \leq s$. Now by definition of c', the only neighbours of v

that may be assigned α by c' are those of $\{v_1, \ldots, v_s\}$. Hence setting $c'(v) = \alpha$, the L-colouring c' is k-improper.

Hence G is k-improper 2-choosable which is a contradiction.

□

Analogously, one can prove the following lemma when $k = 2$.

Lemma 5 *Let D be a discharging digraph of a $(2, 2)$-minimal graph G.*

(i) *The outdegree of a 3-vertex is zero.*

(ii) *If v is an i-vertex with $i \in \{4; 5\}$ then its outdegree is less than i.*

Lemma 6 *Let D be a discharging digraph of a $(1, 2)$-minimal graph G. There is no 3-vertex with outdegree 3 in D.*

Proof. Suppose, for a contradiction, that v is a 3-vertex with outdegree 3. Let u be an outneighbour of v. Let $Q_1 = A^+_D(u)$, $Q_2 = A^+_{D-vu}(v)$, S be the set of leaves of $A^+_D(v)$ with indegree 1 in $A^+_D(v)$ and S' the set of leaves with indegree 2 in $A^+_D(v)$.

Let L be a 2-list-assignment of G. By minimality of G, let c be a 1-improper L-colouring of $G - A^+_D(v)$. Vertices not in S have no neighbour in $G - A^+_D(v)$ and every vertex of S has exactly one neighbour in $G - A^+_D(v)$. Extend c to $S \cup S'$ by assigning to each vertex of S a colour of its list not assigned to its neighbour in $G - A^+_D(v)$ and any colour of its list to a vertex of S'.

Now Q_1 and Q_2 satisfy either (i) or (ii) of Lemma 3. If one of them satisfies (ii), then possibly with recolouring of vertices of S' one can extend c into a 1-improper L-colouring of $G - vu$ such that $c(v) \neq c(u)$. Hence c is a 1-improper L-colouring of G.
If Q_1 and Q_2 both satisfy (i), then possibly with recolouring of vertices of S' one can extend c into a 1-improper L-colouring of $G - vu$ such that $im(v) = im(u) = 0$. Hence c is a 1-improper L-colouring of G.

So G is 1-improper 2-choosable which is a contradiction. □

Proof of Theorem 1. Let G be a $(k, 2)$-minimal graph and D a discharging digraph of G. We start with a charge $w(v) = d(v)$ on each vertex and we apply the following discharging rule: every vertex gives $\frac{k}{k+2}$ to each of its outneighbours.

Let us examine the new charge $w'(v)$ of a vertex v, regarding its degree.

- If v is a 2-vertex, it has indegree 2 so its new charge is $w'(v) = 2 + \frac{2k}{k+2} = \frac{4k+4}{k+2}$.
- If v is a 3-vertex and $k \geq 3$, it has indegree 3 so its new charge is $w'(v) = 3+3\times\frac{k}{k+2} = \frac{6k+6}{k+2} > \frac{4k+4}{k+2}$. If v is a 3-vertex and $k = 2$ then it has outdegree 0 by Lemma 5 and indegree 0 by the construction, and hence $w'(v) = 3$.
- If $4 \leq d(v) \leq k + 1$, $(k \geq 3)$, then by Lemma 4 (i), v has outdegree 0 so its charge is $d(v) \geq 4 > \frac{4k+4}{k+2}$.

- If $k+2 \leq d(v) < \frac{3k}{2}+2$ then either v has outdegree at most $d(v)-2$ and so its new charge is at least $d(v)-(d(v)-2)\times\frac{k}{k+2} = \frac{2d(v)}{k+2}+\frac{2k}{k+2} \geq 2+\frac{2k}{k+2} = \frac{4k+4}{k+2}$, or by Lemmata 4, 5 and 6, it has outdegree $d(v)-1$. In this case, by definition of a discharging digraph, v has indegree 1 so its new charge is:

$$d(v)-(d(v)-1)\times\frac{k}{k+2}+\frac{k}{k+2} = d(v)-(d(v)-2)\times\frac{k}{k+2} \geq \frac{4k+4}{k+2}.$$

- If $\frac{3k}{2}+2 \leq d(v) \leq 2k+1$, $(k \geq 2)$, then by Lemmata 4 and 5, v has outdegree at most $d(v)-1$. So $w'(v) \geq d(v)-(d(v)-1)\times\frac{k}{k+2} = \frac{2d(v)}{k+2}+\frac{k}{k+2} \geq \frac{3k+4+k}{k+2} = \frac{4k+4}{k+2}$.

- If $d(v) \geq 2k+2$, then $w'(v) \geq d(v)(1-\frac{k}{k+2}) = \frac{2d(v)}{k+2} \geq \frac{4k+4}{k+2}$.

Hence $\mathrm{Mad}(G) \geq \frac{1}{|V|}\sum_{v\in V} d(v) = \frac{1}{|V|}\sum_{v\in V} w'(v) \geq \frac{4k+4}{k+2}$. □

2.2 Upper bound for $M(k,2)$

Let us fix $k \geq 1$. In this section, we shall construct a family of graphs $(G_n^k)_{n\geq 1}$ such that for all $n \geq 1$:

- the graph G_n^k is not k-improper 2-colourable; and
- its maximum average degree is $\frac{2n(4k^2+6k+4)+4k^2+6k+2}{2n(k^2+2k+2)+(k+1)^2}$.

This will immediately imply Theorem 2.

We denote by H_k the graph composed of two adjacent vertices u and v also connected by $k+1$ internally disjoint paths of length 2. Take k copies of H_k and create the graph F_k by identifying the vertices v of each copy. Note that F_k has one vertex of degree $k(k+2)$, k vertices of degree $k+2$ and $k(k+1)$ vertices of degree 2. Now take $2n+1$ copies of F_k and join the vertices v of each copy, thereby creating a cycle of size $2n+1$. At last make a subdivision of all the edges of the cycle but one so as to obtain the graph G_n^k.

Lemma 7 *G_n^k is not k-improper 2-colourable.*

Proof. First remark that in any k-improper 2 colouring of H_k, v has impropriety at least 1. Indeed v is a $(k+2)$-vertex in H_k, so if it has impropriety zero then its $k+2$ neighbours are coloured the same, but this is impossible since u is a neighbour of v adjacent to the $k+1$ remaining neighbours. Hence in any k-improper colouring of F_k, v has impropriety k. So in order to colour the whole graph, we must properly colour the subdivided cycle with 2 colours, which is impossible. □

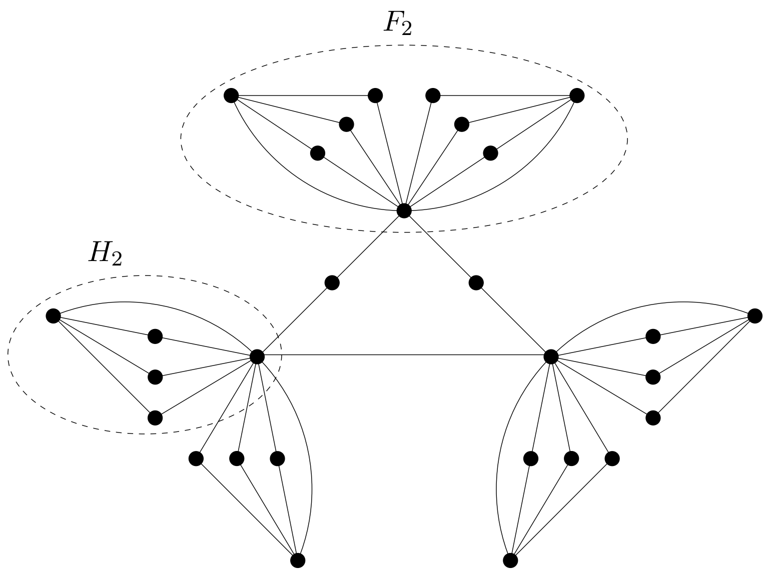

Figure 1: The graph G_1^2

Lemma 8 *The maximum average degree of* G_n^k *is* $M_n^k = \frac{(2n+1)(4k^2+6k+4)-2}{(2n+1)(k^2+2k+2)-1}$.

Proof. As it is easily seen, the maximum average degree of G is its average degree, which is:

$$\frac{(2n+1)[(1 \times k(k+2)+2)+(k \times (k+2))+(k(k+1) \times 2)]+(2n) \times 2}{(2n+1)(1+k+k(k+1))+2n} = M_n^k.$$

□

3 Improper l-choosability, $l \geq 2$

3.1 Lower bound for $M(k,l)$

In this subsection, we shall prove Theorem 3

The result of the theorem is trivial if $k = 0$ since a graph of maximum average degree less than l is $(l-1)$-*degenerate* (i.e. each of its subgraph has a vertex of degree at most $l-1$). Hence it is l-choosable. For bigger values of k, we will need some preliminary results.

Definition 4 A graph is said to be (k,l)-minimal if it is not k-improper l-choosable but each of its proper subgraphs is.

Lemma 9 *Let G be a graph, L a list-assignment and c an L-colouring. If a vertex v has impropriety at least $d(v) - |L(v)| + 2$ under c, then there exists an L-colouring c' of G such that $c'(u) = c(u)$ if $u \neq v$ and $im_{c'}(v) = 0$.*

Proof. Let $c(v) = \alpha$. Then v has at most $d(v) - (d(v) - |L(v)| + 2) = |L(v)| - 2$ neighbours that are not coloured with α. Hence there exists a colour $\beta \in L(v)$ that does not colour any neighbour of v. So setting $c'(v) = \beta$ we obtain the desired colouring. □

We now prove a generalisation of Lemma 1.

Lemma 10 *Let $k \geq 1$ and let G be a (k, l)-minimal graph. The following holds.*

(i) *The minimum degree of G is at least l.*

(ii) *Two $(\leq l + k - 1)$-vertices are not adjacent.*

Proof.

(i) Let L be an l-list-assignment and suppose v is a $(\leq l - 1)$-vertex. By minimality let c be a k-improper L-colouring of $G - v$. As v has at most $l - 1$ neighbours in G, there exists a colour, say α, that is not assigned to any neighbour of v. Hence colouring v with α yields a k-improper L-colouring of G.

Hence G is k-improper l-choosable, a contradiction.

(ii) Let L be an l-list-assignment and suppose, for a contradiction, that u and v are two neighbours of degree at most $l + k - 1$. By minimality, let c be a k-improper L-colouring of $G - \{uv\}$. Then c is an L-colouring of G such that each vertex has impropriety at most k, except possibly u and v which may have impropriety $k + 1$. But in this case we use Lemma 9 to recolour these vertices and obtain a k-improper L-colouring of G.

Hence G is k-improper l-choosable, a contradiction.

□

Definition 5 Let G be a (k, l)-minimal graph. We partially orient G using the following process.

1. Orient each edge uv where v is a $(\leq l + k - 1)$-vertex from u to v.

2. While there is an i-vertex v with $l + k \leq i < l + k + \frac{k}{l}$ having outdegree exactly $i - l + 1$ and indegree 0, we orient one of its unoriented incident edges uv from u to v.

The digraph D induced by the oriented edges is called a *discharging digraph* of G.

The following remark follows from the definition of a discharging digraph.

Remark 1

- *Only vertices of degree less than* $l + k + \frac{k}{l}$ *can have indegree more than zero.*
- *For* $i \leq l + k - 1$*, every* i*-vertex has indegree exactly* i *in* D*.*

Definition 6 A quasi-arborescence rooted at u is a (k,l)-*quasi-arborescence* if and only if:

- every vertex has outdegree at most $\max\{2, 2k-1\}$; and
- every leaf has indegree at most $l + k - 1$.

Now we generalise Lemmata 2 and 3.

Lemma 11 *Let* $k \geq 2$ *and let* Q *be a* (k,l)*-quasi-arborescence rooted at* u*. Let* L *be a list-assignment of* Q *such that* $|L(v)| \geq \max\{1, d_Q(v) - k + 1\}$ *if* v *is a leaf and* $|L(v)| \geq 2$ *otherwise. We denote by* S *the set of leaves that have indegree at least* $k+1$ *in* Q *(and hence a colour-list of size at least 2). Any* L*-colouring of the leaves extends in an* L*-colouring of* Q *such that:*

- *the impropriety of* u *is at most* $k - 1$*; and*
- *for every vertex* $v \notin S, im(v) \leq k$*.*

Furthermore, possibly by recolouring some vertices of S*, this* L*-colouring of* G *can be made* k*-improper.*

Proof. By induction on the number of vertices of Q, the result being trivially true if $|V(Q)| = 1$.

Suppose now that $|V(Q)| > 1$ and the result holds for smaller (k,l)-quasi-arborescences. Let $v_1, \dots, v_s$ be the outneighbours of u in Q. Note that $Q - u$ is the union of s (k,l)-quasi-arborescences Q_i rooted at v_i, $1 \leq i \leq s$, that are disjoint except possibly on their leaves. Let c be an L-colouring of the leaves of Q.

By induction we extend c to an L-colouring of each of the Q_i such that $im(v_i) \leq k-1$. Note that $im_Q(x) = im_{Q_i}(x) \leq k$ for every vertex of Q_i which is not a leaf and $im_Q(x) \leq k$ for each leaf not in S. One of the two colours of $L(u)$, say α, is assigned to at most $k-1$ neighbours of u since $d_Q(u) \leq 2k-1$. Hence setting $c(u) = \alpha$, we obtain the first desired colouring.

Now, we can recolour each leaf f of S with impropriety at least $k+1$ using Lemma 9 since $d_Q(f) - |L(f)| + 2 \leq d_Q(f) - d_Q(f) + k - 1 + 2 = k + 1$. This concludes the proof. □

The above result cannot be extended for $k = 1$. However one can prove the following result.

Lemma 12 *Let Q be a $(1,l)$-quasi-arborescence rooted at u and L a list-assignment of Q such that $|L(v)| \geq 2$ if v is not a leaf, and $|L(v)| \geq d_Q(v)$ otherwise. We denote by S the set of leaves with indegree at least 2. Let c be an L-colouring of the leaves. One of the followings holds.*

(i) *The colouring c can be extended in an L-colouring of Q such that $im(u) = 0$ and $im(v) \leq 1$ if $v \notin S$;*

(ii) *The colouring c can be extended in two different L-colourings of Q c_1 and c_2 such that $c_1(v) = c_2(v)$ if $v \neq u$ and $im^{c_i}(v) \leq 1$ if $v \notin S$.*

Furthermore, possibly by recolouring vertices of S, all these L-colourings can be made 1-improper.
Moreover, if $|L(u)| \geq 3$ then (i) *holds.*

Proof. By induction on the number of vertices, the result being obvious if $|V(Q)| = 1$.

$Q - u$ is the union of two $(1,l)$-quasi-arborescences Q_1 and Q_2 rooted at v_1 and v_2 respectively. They are disjoint except possibly on their leaves. Let c be an L-colouring of the leaves of Q. By induction we extend c to Q_1 and Q_2. Note that for each vertex v of $Q - S$ $im_Q(v) = im_{Q_i}(v) \leq 1$.

If at least one of the Q_i satisfies (ii), or if $|L(u)| \geq 3$, we can suppose that $\{c(v_1), c(v_2)\} \neq L(u)$ and hence we extend c into an L-colouring of Q fulfilling (i).

If both Q_1 and Q_2 satisfy (i), then either $c(v_1) = c(v_2)$ and hence setting $c(u) \in L(u) \setminus \{c(v_1)\}$ yields an L-colouring of Q that satisfies (i); or colouring u with two colours of its list gives the two desired colourings of (ii).

Now we can recolour with impropriety zero each leaf $f \in S$ that has impropriety at least 2 in Q using Lemma 9, since $d_Q(f) - |L(f)| + 2 \leq 2$. This concludes the proof. □

Using these results, we can say more about the structure of a discharging digraph. The following lemma generalises Proposition 2.

Lemma 13 *Let D be a discharging digraph of a (k,l)-minimal graph G.*

(i) *Every vertex u with $l + k \leq d(u) \leq l + 2k - 1$ has outdegree at most $d(u) - l + 1$. In particular, D is acyclic.*

(ii) *For every vertex u with indegree 1, $A^+_D(u)$ is a (k,l)-quasi-arborescence. In particular, the indegree of the leaves in $A^+_D(u)$ is at most $l + k - 1$.*

Proof. (ii) follows easily from (i). So, let us prove (i).

Let L be an l-list-assignment of G. First, D has no 2-circuit since two $(\leq l + k - 1)$-vertices are not adjacent by Lemma 10. Note also that in order to create a circuit in D, it is necessary to create a vertex u of outdegree at least $d(u) - l + 2$. Now suppose, for a contradiction, that D contains a vertex u of outdegree at least $d(u) - l + 2$ and let D' be the digraph obtained just after having created the first such vertex u. Let $u \rightarrow v$ be the last edge that is oriented in D'. u has $d(u)-l+2$ outneighbours (including v) while v has $d(v)-l+1$ outneighbours. We distinguish two cases depending whether the orientation of uv creates a circuit (which is necessary the first), or not.

First Case: the orientation of uv creates a circuit C. Let w be the inneighbour of u in C. We define $Q_1 = A^+_{D'-wu}(v)$, $Q_2 = A^+_{D'-uv}(u)$ and $Q = Q_1 \cup Q_2$. Note that Q_1 and Q_2 are (k,l)-quasi-arborescences which are disjoint, except possibly on some leaves. In particular the outdegree in D' of every internal vertex x of Q is at most $d_G(x)-l+1$. More precisely every internal vertex $x \neq w$ satisfies $d^+_{D'}(x) = d_G(x)-l+1$ while $d^+_Q(w) = d_G(w)-l$ and for every internal vertex x $d^-_{D'}(x) = 1$. Let F be the set of leaves in Q, S the set of leaves that have indegree at least $k+1$ in Q and $\bar{S} = F \setminus S$. We define $\mathring{Q} = Q - \bar{S}$. By minimality, let c be a k-improper L-colouring of $G' = G - \mathring{Q}$. Let $f \in \bar{S}$: if f has impropriety at least $k - d^-_Q(f)+1$, then using Lemma 9 we recolour it with impropriety 0 since $d_{G'}(f)-|L(f)|+2 = d_G(f)-d^-_Q(f)-l+2 \leq l+k-1-d^-_Q(f)-l+2 = k-d^-_Q(f)+1$. Now, let L_1 be the following list-assignment of Q_1:
$L_1(x) = L(x) \setminus \{\alpha — \ \exists z \in N_{G-Q_1}(x), c(z)=\alpha\}$ if $x \notin \bar{S}$, and $L_1(x) = \{c(x)\}$ otherwise. Note that if $x \neq w$ is an internal vertex then:

$$|L_1(x)| \geq l - (d_G(x) - d_{Q_1}(x)) = l - d_G(x) + d_G(x) - l + 1 + 1 = 2$$

and since $d^+(w) = d_G(w) - l$ but u is yet uncoloured:

$$|L_1(w)| \geq l - (d_G(w) - d_{Q_1}(w)) + 1 = l - d_G(w) + d_G(w) - l + 1 + 1 = 2.$$

For the root v, $d^-(v) = 0$ but u is uncoloured yet so:

$$|L_1(v)| \geq l - (d_G(v) - d_{Q_1}(v)) + 1 = l - d_G(v) + d_G(v) - l + 1 + 1 = 2,$$

and for a leaf $f \in S$:

$$|L_1(f)| \geq l - d_G(f) + d_Q(f) \geq l - (l+k-1) + d_Q(f) = d_Q(f) - k + 1.$$

Finally, if $f \in \bar{S}$ then $|L_1(f)| = 1 \geq \max\{1, d_Q(f)-k+1\}$.

Thus we can apply Lemmata 11 and 12. To do so, we L_1-colour all the leaves in Q_1.

Suppose first $k \geq 2$. By Lemma 11, we obtain an L_1-colouring c_1 of Q_1 such that $im^{c_1}_{Q_1}(v) \leq k-1$. Note that c_1 extends c into an L-colouring of $G - Q_2$ such that each vertex has impropriety at most k except possibly some vertices of S. Furthermore, $im_{G-Q_2}(v) \leq k-1$. We define a list-assignment L_2 of Q_2 by $L_2(u) = L(u) \setminus \{\alpha — \ \exists z \neq v \in N_{G-Q_2}(u), c(z)=\alpha)\}$, $L_2(x) = \{c(x)\}$ if x is a leaf and $L_2(x) = L(x) \setminus \{\alpha — \ \exists z \in N_{G-Q_2}(x), c(z)=\alpha\}$ otherwise. Note that we have $|L_2(u)| \geq 2$. We now apply Lemma 11 so as to get an L_2-colouring of Q_2 and hence an L-colouring of G. Every vertex not in $S \cup \{u,v\}$ has impropriety at most k. We have $im_G(u) \leq im_{Q_2}(u)+1 \leq k-1+1 = k$ since there cannot be in $L_2(u)$ the colour of a neighbour of u in $G-(Q_2-v)$. Similarly, $im_G(v) \leq im_{G-Q_2}(v)+1 \leq k$ since the colour of v has been removed from the list of each of its neighbours in $Q_2 - u$. If $f \in S$ has impropriety at least $k+1$, then we recolour it with impropriety 0 using Lemma 9 since $d_G(f) - |L(f)| + 2 \le l + k - 1 - l + 2 = k + 1$. Thus we obtain a k-improper L-colouring of G.

Suppose now $k = 1$. Applying Lemma 12, we obtain an L_1-colouring of $G - Q_2$ such that every vertex not in S has impropriety at most 1, and either v has impropriety 0 (i), or it has impropriety 1 and we can indifferently colour it with two colours of its list (ii). Note that if v has one neighbour distinct from u which is an internal vertex in Q_2 then $|L_1(v)| \ge 3$ so we can suppose that v fulfils (i). Defining L_2 as before, we can apply Lemma 12 to Q_2 so as to obtain an L_2-colouring of Q_2 and hence an L-colouring of G such that u fulfils (i) or (ii). Now, every vertex not in $S \cup \{u, v\}$ has impropriety at most 1. If v satisfies (i), then either u also satisfies (i) or u satisfies (ii) but in this case we can suppose u and v are coloured differently so in all cases they have impropriety at most 1 in G. If v satisfies (ii), then the only neighbour of v in Q_2 is u. Hence we can safely suppose that u and v are coloured differently, so they have impropriety at most 1 in G.

Finally, we can recolour each leaf of S that has impropriety at least 2 by using Lemma 9 and thus we obtain a 1-improper L-colouring of G.

Hence G is k-improper l-choosable, a contradiction.

Second Case: there is no circuit in D'. Then $Q = A^+_{D'}(u)$ is a quasi-arborescence. Moreover each internal vertex v has outdegree at most (and hence exactly) $d(v) - l + 1$. Let $v_1, \dots, v_s$ be the outneighbours of u, we define $Q_j = A^+_{D'}(v_j)$, $1 \le j \le s$. The Q_i are (k, l)-quasi-arborescences that are disjoint except possibly on their leaves. Let F be the set of leaves in Q, S the set of leaves with indegree at least $k + 1$ in Q and $\bar{S} = F \setminus S$. We define $\mathring{Q} = Q - \bar{S}$. Let L be an l-list-assignment of G. By minimality, let c be a k-improper L-colouring of $G' = G - \mathring{Q}$. Let f be a leaf in $\bar{S}$. If f has impropriety at least $k - d_Q(f) + 1$, we recolour it with impropriety 0 using Lemma 9 since: $d_{G'}(f) - |L(f)| + 2 \le d_G(f) - d_Q(f) - l + 2 \le l + k - 1 - d_Q(f) - l + 2 = k - d_Q(f) + 1$.

For each vertex $v \in Q$, we define $L'(v) = L(v) \setminus \{\alpha — \ \exists w \in N_G(v), c(w) = \alpha\}$ if $v \notin \bar{S}$ and $L'(v) = \{c(v)\}$ otherwise. Note that for an internal vertex v:

$$|L'(v)| \ge l - (d_G(v) - d_Q(v)) = l - d_G(v) + d_G(v) - l + 1 + 1 = 2.$$

For a leaf $f \in S$:

$$|L'(f)| \ge l - d_G(f) + d_Q(f) \ge l - (l + k - 1) + d_Q(f) = d_Q(f) - k + 1.$$

Suppose first $k \ge 2$. We L'-colour all the leaves, use Lemma 11 so as to extend it into an L'-colouring of each of the Q_i, and possibly with recolouring some leaves in S we get a k-improper L-colouring of $G - u$ such that $im(v_j) \le k - 1$, $1 \le j \le s$.

Now $|L'(u)| \ge |L(u)| - d(u) + d^+_{D'}(u) = l - d(u) + d(u) - l + 2 \ge 2$. And u has $d^+(u) = d(u) - l + 2 \le 2k + 1$ outneighbours in D'. Thus there is a colour of $L'(u)$, say α, that is assigned to at most k outneighbours of u. Thus setting $c(u) = \alpha$ yields a k-improper L-colouring of G by definition of L'.

Suppose now $k = 1$. We L'-colour all the leaves, use Lemma 12 so as to extend it in an L'-colouring of each of the Q_i, and possibly with recolouring some leaves in S we get a 1-improper L-colouring of $G-u$ such that for each v_j either $im(v_j)=0$ or v_j can safely be recoloured with another colour of $L'(v_j)$.

The same calculation as above shows there exists a colour of $L'(u)$, say α, that is assigned to at most 1 neighbour of u, say v_i. We set $c(u)=\alpha$. If v_i satisfies the first condition, we have a 1-improper L-colouring of G. If v_i satisfies the second condition then we can suppose that $c(u)\neq c(v)$ and thus we also have a 1-improper L-colouring of G.

Hence G is k-improper l-choosable, a contradiction.

□

Proof of Theorem 3. Let G be a (k,l)-minimal graph and D a discharging digraph of G. We start with a charge $w(v)=d(v)$ on each vertex and we apply the following discharging rule: every vertex gives $\frac{k}{l+k}$ to each of its outneighbours.

Let us examine the new charge $w'(v)$ of a vertex v, regarding its degree.

- If $d(v)\le l+k-1$ it has indegree $d(v)$ so its new charge is $w'(v)=d(v)+\frac{d(v)k}{l+k}\ge l+\frac{lk}{l+k}$.
- If $l+k\le d(v)<l+k+\frac{k}{l}$ then either v has outdegree at most $d(v)-l$ and so its new charge is at least $d(v)-(d(v)-l)\times\frac{k}{l+k}=\frac{ld(v)}{l+k}+\frac{lk}{l+k}\ge l+\frac{lk}{l+k}$, or by Lemma 13, it has outdegree $d(v)-l+1$. In this case, by definition of a discharging digraph, v has indegree 1 so its new charge is:

$$w'(v)=d(v)-(d(v)-l+1)\times\frac{k}{l+k}+\frac{k}{l+k}=d(v)-(d(v)-l)\times\frac{k}{l+k}\ge l+\frac{lk}{l+k}.$$

- If $l+k+\frac{k}{l}\le d(v)\le l+2k-1$, then by Lemma 13, v has outdegree at most $d(v)-l+1$. So $w'(v)\ge d(v)-(d(v)-l+1)\times\frac{k}{l+k}=\frac{ld(v)}{l+k}+\frac{kl-k}{l+k}\ge\frac{l^2+2kl}{l+k}=l+\frac{kl}{l+k}$.
- If $d(v)\ge l+2k$, then $w'(v)\ge d(v)(1-\frac{k}{l+k})=\frac{ld(v)}{l+k}\ge\frac{l^2+2kl}{l+k}=l+\frac{kl}{l+k}$.

Hence $\text{Mad}(G)\ge\frac{1}{|V|}\sum_{v\in V}d(v)=\frac{1}{|V|}\sum_{v\in V}w'(v)\ge l+\frac{kl}{l+k}$.

□

3.2 Upper bound for $M(k,l)$

In this subsection we shall construct, for every $l\ge 2$ and every $k\ge 1$, a graph G_l^k which is not k-improper l-colourable. So its maximum average degree will give an upper bound for $M(k,l)$. To construct G_2^k, take $k+1$ copies of H_k (defined in Subsection 2.2) and identify their vertex v. We define G_l^k, $l\ge 3$, inductively. First, create the graph M_l^k by taking k copies of G_{l-1}^k and adding a vertex w linked to all the other vertices. Then take

$l-1$ copies $M^1, \ldots, M^{l-1}$ of M_l^k and join all the vertices $w_1, \ldots, w_{l-1}$ (so that they form a complete graph of size $l-1$). Now, add $k+2$ vertices $z_0, z_1, \ldots, z_{k+1}$ each joined to each w_i for $i \in \{1, \ldots, l-1\}$. Last, add the edges $z_0 z_i$ for $i \in \{1, \ldots, k+1\}$.

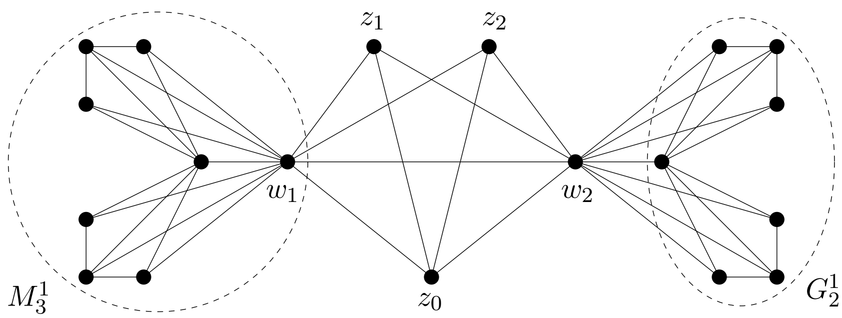

Figure 2: The graph G_3^1

Lemma 14 *For every $l \geq 2$ and every $k \geq 1$, the graph G_l^k is not k-improper l-colourable.*

Proof. The result is clear for G_2^k. Suppose the result is true for $l-1 \geq 2$ and let us prove it for G_l^k. First note that in any k-improper l-colouring of M^i, the vertex w_i has impropriety k. Indeed, w_i has a neighbour of its colour in each copy of G_{l-1}^k since otherwise G_{l-1}^k would be k-improper $(l-1)$-colourable. Hence each of the w_i, $1 \leq i \leq l-1$, cannot have any neighbour of its colour in $G_l^k - M^i$. In particular, as the subgraph induced by $w_1, \ldots w_{l-1}$ is complete, all the z_i, $0 \leq i \leq k+1$, must be coloured the same. But then z_0 must have impropriety $k+1$. $\square$

Lemma 15 *The number of vertices of G_l^k is:*

$$n_l^k := 2l + (l+1)k + \sum_{i=2}^{l} \frac{(l-1)!}{(l-i)!} k^i.$$

In particular, as a polynomial in k $n_l^k \sim (l-1)!k^l$.

Proof. We have $n_2^k = k^2 + 3k + 3$ and $\forall l \geq 3, n_l^k = (k \times n_{l-1}^k + 1) \times (l-1) + k + 2$. $\square$

Lemma 16 *The sum of the degrees of the vertices in G_l^k, denoted by s_l^k, is a polynomial in k such that $s_l^k \sim 2l!k^l$.*

Proof. We have $s_2^k = 4k^2 + 10k + 6$ and $s_l^k = (l-1)(k \times s_{l-1}^k + 2k \times n_{l-1}^k + l + k) + (l+1)k + 2l$ if $l \geq 3$. Hence s_l^k is a polynomial in k of degree l. Furthermore, denoting by c_l^k its dominant coefficient, we have $c_2^k = 4$ and $\forall l \geq 3, c_l^k = (l-1) \times c_{l-1}^k + 2k \times (l-1)!$. Thus $c_l^k = 2l!$. $\square$

Proposition 3 *For every fixed $l \geq 2$, the maximum average degree of G_l^k tends to $2l$ as k tends to infinity.*

Proof. It is clear that the maximum average degree of G_l^k is its average degree. Then by Lemmata 15 and 16, we have

$$\lim_{k\to\infty} \mathrm{Mad}(G_l^k) = 2\frac{l!}{(l-1)!} = 2l.$$

□

Corollary 1 immediately follows from Theorem 3 and Proposition 3.

3.3 Application to graphs of higher genus

Using the general Euler formula, one can prove that for any graph H of genus r and girth at least g

$$\mathrm{Ad}(H) \leq \frac{2g}{g-2} + \frac{4g(r-1)}{(g-2)|V(H)|}.$$

This contributes to obtain the same results for toroidal graphs (that is graphs of genus 1). In particular, this yields the following interesting corollary.

Corollary 3 *Every triangle-free toroidal graph is* 1*-improper* 4*-choosable.*

Proof. The maximum average degree of a triangle-free toroidal graph is at most $\frac{2\times 4}{4-2} = 4 < \frac{24}{5} = 4 + \frac{4\times 1}{4+1}$. □

Note that it is still unknown whether or not all toroidal graphs are 1-improper 4-colourable.

We now pay attention to graphs of genus at least 2. Miao [5] proved the following for every graph G of genus $r \geq 2$:

- if G has at least 21 faces and girth at least $r + 9$ then G is 1-improper 2-choosable;
- if G has at least 13 faces and girth at least $\lceil \frac{6}{5}(r + 5)\rceil$ then G is 2-improper 2-choosable;
- if G has at least 14 faces and girth at least $r + 5$ then G is 3-improper 2-choosable.

We shall show now that our results improve these ones. For any g, let $f(\mathrm{Ad}, g) = 2\sum_{i=0}^{s-1}(\mathrm{Ad}-1)^i$ if $g = 2s$ and $f(\mathrm{Ad}, g) = 1 + \mathrm{Ad}\sum_{i=0}^{s-1}(\mathrm{Ad}-1)^i$ if $g = 2s+1$. It has been proved in [6] that any graph H of girth g satisfies $|V(H)| \geq f(\mathrm{Ad}, g)$. Hence, we deduce that the maximum average degree of any graph G of genus r and girth g satisfies

$$f(\mathrm{Mad}(G), g)(\mathrm{Mad}(G) - \frac{2g}{g-2}) - \frac{4(r-1)g}{g-2} \leq 0.$$

Thus we can easily deduce conditions on girth and genus that allow to use Theorem 3. For instance, we obtain the following corollary.

Corollary 4 *Let G be a graph of genus $r \geq 2$.*

- *If G has girth at least $r + 8$ then G is 1-improper 2-choosable.*
- *If G has girth at least $r + 6$ then G is 2-improper 2-choosable.*
- *If G has girth at least $r + 5$ then G is 3-improper 2-choosable.*

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