Projective, affine, and abelian colorings of cubic graphs

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Abstract

We develop an idea of a *local* 3-*edge-coloring* of a cubic graph, a generalization of the usual 3-edge-coloring. We allow for an unlimited number of colors but require that the colors of two edges meeting at a vertex always determine the same third color. Local 3-edge-colorings are described in terms of colorings by points of a partial Steiner triple system such that the colors meeting at each vertex form a triple of the system. An important place in our investigation is held by the two smallest non-trivial Steiner triple systems, the Fano plane $PG(2,2)$ and the affine plane $AG(2,3)$. For $i = 4, 5$, and 6 we identify certain configurations F_i and A_i of i lines of the Fano plane and the affine plane, respectively, and prove a theorem saying that a cubic graph admits an F_i-coloring if and only if it admits an A_i-coloring.

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Among consequences of this is the result of Holroyd and Škoviera (2004) that the edges of every bridgeless cubic graph can be colored by using points and blocks of any non-trivial Steiner triple system S. Another consequence is that every bridgeless cubic graph has a proper edge-coloring by elements of any abelian group of order at least 12 such that around each vertex the group elements sum to 0.

We also propose several conjectures concerning edge-coloring of cubic graphs and relate them to several well-known conjectures. In particular, we show that both the Cycle Double Cover Conjecture and the Fulkerson Conjecture can be formulated as a coloring problem in terms of known geometric configurations—the Desargues configuration and the Cremona-Richmond configuration, respectively.

1 Introduction

Edge colorings of cubic graphs have been extensively studied for more than a century. The original incentive came in 1880 from Tait's attempt to solve the Four Color Problem [33], and during the subsequent decades this concept has established close connections to other areas of graph theory, including nowhere-zero flows and embeddings of graphs on surfaces.

Edge-colorings divide cubic graphs into two uneven parts. The class of 3-edge-colorable graphs comprises almost all cubic graphs (Robinson and Wormald [31]) and seems to be easier to understand. Its complement is an extremely sparse class of graphs consisting of graphs with chromatic index four and reputed for being closely related to several difficult problems in graph theory. "Non-trivial" members of this family are known as *snarks* and may include counterexamples to the Cycle Double Cover Conjecture, the Five Flow Conjecture, and Fulkerson's Conjecture.

The classification problem, i.e., the problem of determining whether a cubic graph has chromatic index three or four, is very interesting but, as Holyer [19] showed, is exceedingly difficult. It is therefore surprising that very little attention has so far been given to generalizations of classical 3-edge-colorings. Such generalizations might shed new light on the classification problem and on several other problems related to edge-colorings of graphs.

A natural way to generalize the concept of a 3-edge-coloring is to replace the global condition on the number of colors by a local one. This can be done, for instance, by allowing the number of colors to be arbitrary, but requiring that any two colors meeting at a vertex always determine the same third color. This condition is automatically fulfilled whenever only three colors are used. Therefore, such colorings include 3-edge-colorings as a special case.

Our local condition allows us to regard the colors as points of a Steiner triple system $\mathcal{S}$, with triples of colors occurring at vertices being blocks of the system. This is because in a Steiner triple system any two points belong to exactly one block. Of course, such a coloring (called a Steiner coloring, or more specifically, an $\mathcal{S}$-coloring) need not use up all the points or all the blocks of the system. Thus, in general, it is more appropriate to speak of edge-colorings by *partial* Steiner triple systems, or equivalently, by configurations of points and blocks contained in Steiner triple systems.

Steiner colorings have been previously considered by several authors. In 1986, Archdeacon [1, 2] proposed the study of general Steiner colorings and conjectured that every bridgeless cubic graph admits an $\mathcal{S}$-coloring for each Steiner triple system $\mathcal{S}$ of order greater than three. He also observed that every bridgeless cubic graph has a coloring by the smallest non-trivial Steiner triple system, the projective plane $PG(2,2)$ with 7 points known as the Fano plane F_7 (see the left part of Figure 1). In 2004, Holroyd and Škoviera [18] confirmed Archdeacon's conjecture. Their proof identified an "unavoidable set" $\mathbf{U}$ of three configurations (shown in Figure 1) such that

(i) every non-trivial Steiner triple system contains at least one member of $\mathbf{U}$; and

(ii) each configuration in $\mathbf{U}$ colors every bridgeless cubic graph.

The geometric structure of Fano colorings was subsequently investigated by Máčajová and Škoviera [26]. They showed that six (and conjectured that four) lines of the Fano plane covering all seven points are enough to color every bridgeless cubic graph. They also proved that their Four-Line Conjecture is equivalent to an older conjecture of Fan and Raspaud [12]: *every bridgeless cubic graph has three perfect matchings with empty intersection.* The equivalence of these two conjectures establishes a connection between Steiner colorings and other areas of graph theory such as cycle coverings of graphs or Fulkerson's conjecture.

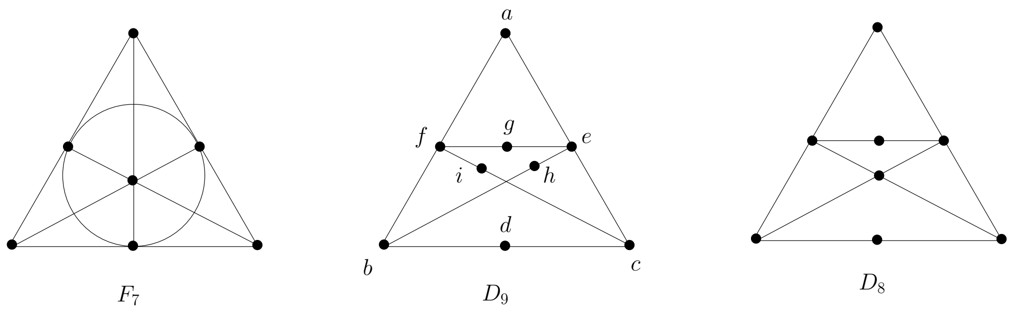

Figure 1: Unavoidable set of configurations for non-trivial Steiner triple systems

Archdeacon [1] also proposed another generalization of 3-edge-colorings of cubic graphs. Given a finite abelian group A, an A-coloring of a cubic graph G is an assignment of non-zero elements of A to the edges of G subject to the condition that for each vertex v the values on the edges incident with v sum to 0 in A. Note that the elements assigned to incident edges do not need to be distinct. This concept is an undirected analogue of nowhere-zero A-flows and, at the same time, a generalization of 3-edge-colorings since a $\mathbb{Z}_2 \times \mathbb{Z}_2$-coloring is nothing but the usual 3-edge-coloring.

To emphasize the exceptional role of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, Archdeacon [1] conjectured that an A-coloring exists for each bridgeless cubic graph and each abelian group A of order at least five. This conjecture was settled by Máčajová et al. [27] by exploiting the fact that, in contrast to $\mathbb{Z}_2 \times \mathbb{Z}_2$-colorings, general abelian colorings need not be proper, i.e., incident edges can be assigned the same color.

Proper abelian colorings generalize 3-edge-colorings. It transpires that abelian colorings can be conveniently studied within the context of partial Steiner colorings. For every abelian group A, one can define a partial Steiner triple system $\mathcal{C}(A)$ whose points are all non-zero elements of A and blocks are all 3-element subsets of $A - \{0\}$ with zero sum. Thus, $\mathcal{C}(A)$-colorings coincide with proper A-colorings.

By employing this interpretation, Máčajová et al. [27] noticed that there are groups that do not color all bridgeless cubic graphs (e.g., cyclic groups of order smaller than 10) and they sketched a proof of the fact that all abelian groups of order at least 12 do. As for the four groups $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_{10}$ and $\mathbb{Z}_{11}$, the existence of proper colorings remains open. Each of these groups contains a configuration of four lines of the Fano plane covering all seven points, so the existence of such colorings would follow from the Four-Line Conjecture.

In the present paper we continue the study of abelian colorings but with different emphasis. Instead of treating abelian colorings directly, we focus on relationships between Steiner colorings hidden below the surface of abelian colorings. Two particular Steiner triple systems play a prominent role in our analysis, the projective plane $PG(2,2)$ with 7 points (the Fano plane), and the affine plane $A(2,3)$ with 9 points. Our main result, Theorem 3.2, shows that for $i \in \{4,5,6\}$ each of these systems contains a configuration of i lines, denoted by F_i and A_i, respectively (see Figure 4), such that a cubic graph is F_i-colorable if and only if it is A_i-colorable. This equivalence is rather surprising as these colorings are based on projective and affine geometries over fields of coprime characteristic.

Theorem 3.2 has several important consequences. First of all, the fact mentioned above that every bridgeless cubic graph has an F_6-coloring [26] now implies that it also has an A_6-coloring. Since there is a copy of F_6 or a copy of A_6 in $\mathcal{C}(A)$ for every abelian group A of order at least 12, it follows that every bridgeless cubic graph has a proper A-coloring for each such group.

As regards the four exceptional groups $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_{10}$, and $\mathbb{Z}_{11}$, we note that $\mathcal{C}(\mathbb{Z}_4 \times \mathbb{Z}_2)$ coincides with F_5 (see Figure 6). Although this configuration is not contained in other exceptional groups, the "equivalent" configuration A_5 is. It follows that the existence of a proper $\mathbb{Z}_4 \times \mathbb{Z}_2$-coloring implies the existence of a proper coloring by each of the remaining exceptional groups.

In addition, we can easily deduce the main result of Holroyd and Škoviera [18], asserting that every bridgeless cubic graph has an S-coloring for every non-trivial Steiner triple system S. Indeed, it directly follows from Theorem 3.2 that every bridgeless cubic graph admits a coloring by each member of the unavoidable set $\mathbf{U}$ depicted in Figure 1.

The paper is organized as follows. In the next section we deal with several topics related to edge-colorings and partial Steiner triple systems that we need throughout the paper. In particular, we show that F_4, which is also called the *sail configuration*, is the smallest configuration that could color every bridgeless cubic graph and state three related conjectures. Section 3 is devoted to the main result of this paper, Theorem 3.2, and its proof. The next three sections deal with applications of Theorem 3.2 to general Steiner colorings, to abelian colorings and to various modifications. In the final section we return to colorings by configurations in the general sense and concentrate on so-called symmetric configurations. We show that three well-known conjectures, the Cycle Double Cover Conjecture, the Fulkerson Conjecture and the Petersen Coloring Conjecture can all be formulated as coloring problems in terms of symmetric point-line configurations such as the Desargues configuration and the Cremona-Richmond configuration known from geometry.

2 Colorings and configurations

Graphs considered in this paper are finite, with parallel edges and loops permitted. For the most part, however, they are cubic and loopless, as edge-colorings exclude loops. From now on, an *edge-coloring* of a graph is an assignment of colors to the edges of a graph in such a way that adjacent edges receive distinct colors. Our aim is to study edge-colorings of cubic graphs where the set of colors is endowed with the structure of a partial Steiner triple system subject to the condition that the colors meeting at a vertex form a triple of the system.

A *Steiner triple system* $\mathcal{S} = (P, B)$ of order n is a collection B of three-element subsets (called *triples* or *blocks*) of a set P of n *points* such that each pair of points is together present in *exactly* one triple. The smallest Steiner triple system is the *trivial system* I which has three points and a single block. In general, a Steiner triple system of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$ (see, e.g., the monograph by Colbourn and Rosa [6]).

If each pair of points is contained in *at most* one triple, and if there are no isolated points, we say that $\mathcal{S}$ is a *partial Steiner triple system*. Note that there is a partial Steiner triple system of order n for each $n \geq 5$.

As shown by Treash [34] in 1971, every partial Steiner triple system can be embedded into a full Steiner triple system (see also [6]). A partial Steiner triple system can thus be thought of as a configuration of points and blocks of a Steiner triple system. This justifies the term *configuration* which we use as a short synonym for partial Steiner triple system. (Our usage follows the one of Grannell et al. [14, 15] and differs from the one of Gropp [16, 17].)

It is sometimes helpful to transform a partial Steiner triple system into another one. This can be done by mapping the points of $\mathcal{S}$ to points of $\mathcal{T}$ in such a way that each block of $\mathcal{S}$ becomes a block of $\mathcal{T}$. Such a mapping is called a *homomorphism* from $\mathcal{S}$ to $\mathcal{T}$ and is denoted by $\mathcal{S} \to \mathcal{T}$. Note that a homomorphism is not necessarily injective, but it must be injective on each block. If $\mathcal{S} \to \mathcal{T}$, we usually say that $\mathcal{S}$ *maps* to $\mathcal{T}$.

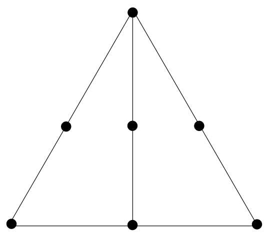

Figure 2: The smallest class 2 configuration $C_{15} \cong F_4$

Many interesting partial Steiner triple systems come from geometrical configurations. Two important examples are the projective and the affine Steiner triple systems. For $n \geq 2$, the *projective* Steiner triple system $PG(n, 2)$ has $\mathbb{Z}_2^{n+1} - \{0\}$ as its point set, the blocks of the system being the triples $\{x, y, z\}$ of points such that $x + y + z = 0$. For $n \geq 2$, the *affine* Steiner triple system $AG(n, 3)$ has point set $\mathbb{Z}_3^n$, the triples of the system being again the triples of distinct points with zero sum. The first of these classes includes the smallest non-trivial Steiner triple system $PG(2, 2)$, the *Fano plane*, which has 7 points. The second smallest non-trivial Steiner triple system is the unique system with 9 points, the *affine plane* $AG(2, 3)$.

Certain projective and affine configurations will play an important role in our further study. For example, it is well known that the Fano plane has two non-isomorphic configurations of four lines: C_{15} on seven points (see Figure 2), and the *Pasch configuration* C_{16} isomorphic to the Fano plane minus a point (we follow the notation used by Grannell et al. [14, 15]). The Pasch configuration is the only partial Steiner triple system with six points and four blocks. In case of seven point configurations contained in the Fano plane, we define F_m to be the unique configuration isomorphic to m lines of the Fano plane covering all seven points, for each $m \in \{4, \ldots, 7\}$. (See Figure 4.)

The affine plane $AG(2, 3)$ contains two non-isomorphic configurations of four lines and seven points: C_{14} shown in Figure 5 and $C_{15} \cong F_4$. In the context of the affine plane, the latter configuration will be denoted by A_4. The configuration A_4 can be extended into a

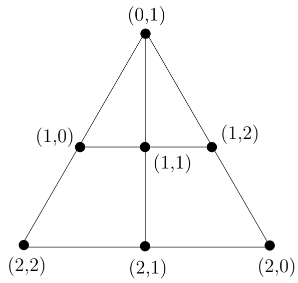

Figure 3: The mitre configuration along with affine coordinates of its points

five-line configuration of $AG(2,3)$ in two different ways. If the new line entirely consists of points of A_4, the resulting configuration is called the *mitre* (it is shown in Figure 3). Otherwise, the configuration has eight points and is denoted by A_5 (see Figure 4). Among the seven non-isomorphic configurations of six lines covering all nine points of $AG(2,3)$ we deal only with the configuration $A_6 \cong D_9$ displayed in Figures 1 and 4. For more information about Steiner triple systems and configurations the reader may consult the monograph by Colbourn and Rosa [6].

Let us now return to colorings. Given a partial Steiner triple system $\mathcal{S}$, an $\mathcal{S}$-*coloring* of a cubic graph G is a coloring of the edges of G by points of $\mathcal{S}$ such that the colors of any three pairwise incident edges form a block of $\mathcal{S}$. A graph which admits such a coloring is said to be $\mathcal{S}$-*colorable.* If a cubic graph G is $\mathcal{S}$-colorable and $\mathcal{S}$ maps to a configuration $\mathcal{T}$, then G is also $\mathcal{T}$-colorable. In particular, if $\mathcal{S}$ maps to the trivial system I, then a cubic graph is $\mathcal{S}$-colorable if and only if it is 3-edge-colorable. Borrowing our terminology from Vizing's edge-coloring theorem, we call a non-empty configuration *class 1* if it maps to I, and *class 2* otherwise. For example, C_{14} and the Pasch configuration C_{16} are easily checked to be class 1 whereas $C_{15} \cong F_4$ is class 2. The latter can either be verified directly or can be derived from the fact that the Petersen graph is F_4-colorable [26, Figure 1] but not 3-edge-colorable.

In fact, F_4 is the smallest class 2 configuration. We leave the straightforward proof of the following proposition to the reader.

Proposition 2.1. *Let $\mathcal{C}$ be a configuration of class* 2 *with the least number of points and blocks. Then $\mathcal{C}$ is isomorphic to F_4.*

Somewhat surprisingly, the smallest class 2 configuration F_4 seems to be sufficient to color every bridgeless cubic graph. Indeed, no bridgeless cubic graph that lacks an F_4-coloring has been found so far. This led Máčajová and Škoviera [26] to propose the following conjecture.

Conjecture 2.2. (Four-Line Conjecture) *Every bridgeless cubic graph admits an F_4-coloring.*

Danziger et al. [10] showed that the F_5-configuration of the Fano plane (known as *mia*) and the mitre are the only two five-line configurations on seven points. Both of them contain the four-line configuration of the Fano plane F_4, and so these three configurations are the smallest three configurations of class 2. Therefore the following two conjectures [26] are natural relaxations of the Four-Line Conjecture.

Conjecture 2.3. (Five-Line Conjecture) *Every bridgeless cubic graph admits an F_5-coloring.*

Conjecture 2.4. (Mitre Conjecture) *Every bridgeless cubic graph admits a mitre-coloring.*

We point out that Kaiser and Raspaud [25] have recently verified the 5-Line Conjecture for bridgeless cubic graphs of oddness 2 (the *oddness* of a cubic graph G being the minimum number of odd circuits in a 2-factor of G).

Colorings by projective configurations can conveniently be seen as nowhere-zero flows. An *A-flow* on a graph G is an orientation of the edges of G and a function $\xi\colon E(G) \to A$ from the edge-set of G to an abelian group A (written additively) such that for each vertex the sum of incoming values equals the sum of outgoing values. A flow is *nowhere-zero* if it is non-zero on every edge of G.

If each element of A is self-opposite, then the orientation of G becomes irrelevant and we may view ξ as a function on an undirected rather than a directed graph. In this case, the group A is isomorphic to a direct product of copies of $\mathbb{Z}_2$.

Since the lines of any projective Steiner triple system correspond to triples of points from $\mathbb{Z}_2^{n+1} - \{0\}$ whose sum is 0, it follows immediately from the definition that a coloring by any configuration contained in a projective Steiner triple system $PG(n, 2)$ is just a nowhere-zero $\mathbb{Z}_2^{n+1}$-flow on G. An important consequence of this fact is that a cubic graph which has a bridge cannot be colored by any projective configuration because an arbitrary flow must take the value zero on any bridge. Conversely, every bridgeless cubic graph G admits a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-flow (see [11, Chapter 6], or [23]), and hence G can be F_7-colored. Thus a cubic graph is F_7-colorable if and only if it is bridgeless.

3 Projective and affine colorings

The purpose of this section is to establish a fundamental relationship between the projective and the affine colorings of a cubic graph, more precisely between the colorings by configurations in the Fano plane and the colorings by configurations in the affine plane.

Let us start with the observation that for $i = 4, 5$ and 6 the projective configuration F_i is a homomorphic image of the affine configuration A_i. This is trivial for $i = 4$ because $F_4 \cong A_4$. Furthermore, F_5 arises from A_5 by identifying the points labeled $(0, 1)$ and $(0, 2)$

into one point (see the middle part of Figure 4), and F_6 results from $A_6 \cong D_9$ by identifying the points labeled $(0,1)$ and $(0,2)$ into one point and the points labeled $(1,0)$ and $(2,0)$ into another point (see Figure 4). In all the three cases the identified pairs of points come from disjoint blocks, implying that the resulting mapping is a homomorphism. Thus A_i maps onto F_i and, consequently, every A_i-coloring yields an F_i-coloring.

Surprisingly, there is a relationship between the colorings in the opposite direction, too. This relationship is far less obvious because it cannot be supported by a homomorphism argument. Nevertheless, we show that each F_i-coloring of a cubic graph G gives rise to an A_i-coloring of G, although the resulting coloring need not be uniquely determined by an F_i-coloring anymore.

The tool that transfers the colorings from the Fano plane to the affine plane involves the structural concept of a "triad" of parity subgraphs. Following Zhang [35], we define a *parity subgraph* of a graph G to be a subgraph P with the property that for each vertex v of G the degree of v in P has the same parity as its degree in G.

In a cubic graph every parity subgraph is a spanning subgraph with all vertices having degree one or three. We may unambiguously identify such a parity subgraph with its edge-set. A *triad* of a cubic graph G is a set $\{P_1, P_2, P_3\}$ of three parity subgraphs of G such that $P_1 \cap P_2 \cap P_3 = \emptyset$. Note that a cubic graph containing a triad must be bridgeless because a bridge belongs to every parity subgraph.

In a cubic graph, each 1-factor is a parity subgraph. Let us call the number of 1-factors in a triad its *weight*. The weight then measures the "quality" of a triad—the heavier a triad is, the more difficult it is to find.

It may be useful to note that the concept of a parity subgraph is in some sense complementary to the concept of a $\mathbb{Z}_2$-flow on a graph. Indeed, the complement $G - E(P)$ of a parity subgraph P is an *even subgraph* of G, i.e., a spanning subgraph with all vertices of even degree. In turn, every even subgraph H corresponds to a unique $\mathbb{Z}_2$-flow: an edge of G belongs to H if and only if its flow value is 1. Thus, we can say that a given even subgraph *determines* a $\mathbb{Z}_2$-flow, or that an even subgraph is *determined* by a given $\mathbb{Z}_2$-flow. Hence for every graph there is a one-to-one correspondence between its parity subgraphs and its $\mathbb{Z}_2$-flows.

In particular, if $\{P_1, P_2, P_3\}$ is a triad in a cubic graph G, then the set $\{P_1', P_2', P_3'\}$ consisting of the complements $P_i' = E(G) - P_i$, is a covering of G by three even subgraphs. The weight of the triad then equals the number of 2-factors in $\{P_1', P_2', P_3'\}$.

Our first result shows that triads of parity subgraphs in a cubic graph are essentially F_i-colorings.

Theorem 3.1. *In every cubic graph there exists a one-to-one correspondence between the triads of weight w and the F_{7-w}-colorings.*

Proof. Let G be a cubic graph, and let $\{P_1, P_2, P_3\}$ be a triad of weight w in G where $0 \le w \le 3$. We may assume that 1-factors of the triad are listed first. We show that G can be F_{7-w}-colored. Define a mapping $\phi : E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by setting $\phi(e) = (\phi_1(e), \phi_2(e), \phi_3(e))$ where $\phi_i(e) = 0$ if and only if the edge e belongs to P_i. Observe that each coordinate mapping ϕ_i is the characteristic function of an even subgraph. Hence ϕ_i is a $\mathbb{Z}_2$-flow. As $P_1 \cap P_2 \cap P_3 = \emptyset$, no edge receives the value $(0,0,0)$ by ϕ. By a direct verification one can easily see that this coloring does not use the first w of the following lines in the Fano plane: $l_1 = \{(0,0,1),(0,1,0),(0,1,1)\}$, $l_2 = \{(0,0,1),(1,0,0),(1,0,1)\}$, $l_3 = \{(0,1,0),(1,0,0),(1,1,0)\}$. For example, if P_1 is a 1-factor, then at each vertex of G the colors of exactly two edges have their first coordinate equal to 1. This excludes the line l_1, but not l_2 and l_3. The situation is similar for P_2 and P_3. Finally, it follows from the definition that two distinct triads result in two distinct F_{7-w}-colorings.

Let, on the other hand, $\phi = (\phi_1, \phi_2, \phi_3)$ be a F_{7-w}-coloring which omits the first w of the lines l_1, l_2, and l_3 described above. For $i = 1, 2, 3$ define P_i to be the spanning subgraph formed by the set of all edges e for which $\phi_i(e) = 0$. Since each ϕ_i is a $\mathbb{Z}_2$-flow on G, the subgraph P_i is a complement of an even subgraph and therefore a parity subgraph of G. As the triple $(0,0,0)$ does not occur in the Fano plane, we have $P_1 \cap P_2 \cap P_3 = \emptyset$. Thus $\{P_1, P_2, P_3\}$ forms a triad.

Note that the Fano plane contains exactly one line with 0 on the i-th coordinate of all its three points, namely the line $\mathbf{x}_i = 0$ which is exactly the line l_i. Therefore, P_i is a 1-factor only if the line l_i is not used in the coloring. It follows that the weight of $\{P_1, P_2, P_3\}$ equals w. □

Theorem 3.1 with $m = 3$ implies that Conjecture 2.2 is equivalent to an older conjecture of Fan and Raspaud [12] asserting that in every bridgeless cubic graph there exist three perfect matchings with no edge in common. This equivalence was first proved by Máčajová and Škoviera [26].

Before proceeding with the main result we introduce another important tool. Given a graph G and a spanning subgraph $H \subseteq G$, we define the *quotient graph* G/H of G by H to be the graph obtained from G by contracting each component of H into a single vertex. In addition, for any spanning subgraph K of G we set K/H to be the subgraph $(K \cup H)/H$ of G/H. Note that in general the quotient graph K/H may have multiple edges and loops even when K is simple.

By using a straightforward flow argument one can establish the following useful property of the quotient mapping $G \to G/H$:

If P is a parity subgraph of G, then P/H is a parity subgraph of G/H.

We are now ready for the main result.

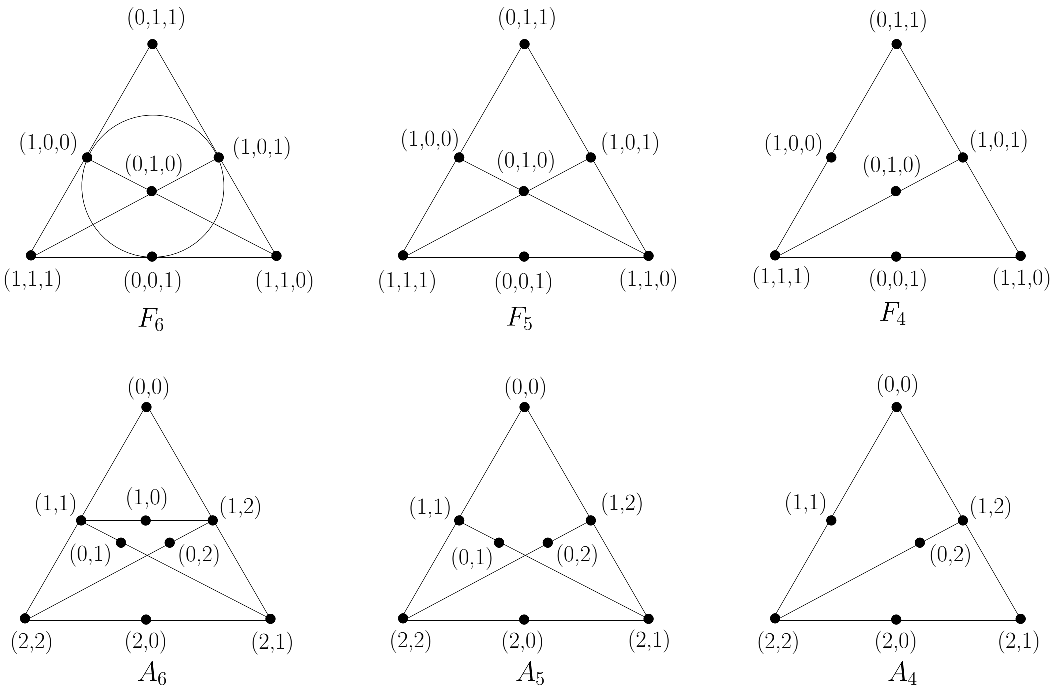

Figure 4: Projective and affine configurations in Theorem 3.2

Theorem 3.2. *Let G be a bridgeless cubic graph and let $i \in \{4, 5, 6\}$. Then G admits an F_i-coloring if and only if it admits an A_i-coloring.*

Proof. An A_i-coloring induces an F_i-coloring for each $i \in \{4, 5, 6\}$ because the configuration F_i is a homomorphic image of the configuration A_i. For the converse, we use a method similar to that of Holroyd and Škoviera [18, Lemma 5.2]. Assume that a cubic graph G has an F_i-coloring for some $i \in \{4, 5, 6\}$. We want to show that G also has an A_i-coloring. By Theorem 3.1, G contains a triad $\{P_1, P_2, P_3\}$ of weight $w = 7 - i$. We may assume that 1-factors are listed first in the triad. In particular, the parity subgraph P_1 is a 1-factor.

Before constructing an A_i-coloring of G, we modify the triad $\{P_1, P_2, P_3\}$ to obtain a new triad $\{Q_1, Q_2, Q_3\}$ with a more convenient structure. Let F be the 2-factor of G complementary to P_1. Let $j \in \{1, 2, 3\}$. If P_j is a 1-factor, set $Q_j = P_j$. If P_j is not a 1-factor, we proceed as follows. Since P_j is a parity subgraph of G, the quotient P_j/F is a parity subgraph of G/F. Observe that every graph contains an acyclic parity subgraph. Let P'_j be an acyclic parity subgraph of P_j/F. There exists a parity subgraph Q_j of G such that $Q_j \cap P_1 = P'_j$. Indeed, we construct Q_j by first setting $Q_j := P'_j$. Next, we add some edges of F to Q_j as follows. For each circuit $C := v_1v_2 \dots v_n$ of the 2-factor F, we proceed around the circuit from v_1 to v_n, and we add the edge v_iv_{i+1} to Q_j if v_i is incident to 0 or 2 edges of Q_j (the indices are taken modulo n). The fact that we obtain a parity subgraph of G follows from the fact that P'_j is a parity subgraph of G/F, hence the number of edges of P'_j leaving C has the same parity as the length n of C. Since $Q_1 \cap Q_j = P_1 \cap Q_j \subseteq P_1 \cap P'_j$, it follows that $Q_1 \cap Q_2 \cap Q_3 = \emptyset$.

In order to derive an affine coloring from $\{Q_1, Q_2, Q_3\}$, we define a *weak* 3*-edge-coloring* of a cubic graph K to be a mapping $\theta : E(K) \to \mathbb{Z}_3$ such that, at each vertex of K, the colors are either all distinct or all equal. Furthermore the *weakness set* of θ is the set of those vertices of G where the colors are all equal. It is straightforward to see that a mapping $\theta = (\theta_1, \theta_2) : E(K) \to \mathbb{Z}_3 \times \mathbb{Z}_3$ is an $AG(2, 3)$-coloring of K if and only if both θ_1 and θ_2 are weak 3-edge-colorings and their weakness sets are disjoint.

For each $j \in \{2, 3\}$, we use Q_j to define a weak 3-edge-coloring $\psi_j : E(G) \to \mathbb{Z}_3$.

If Q_j is a 1-factor, color the edges of $P_1 \cap Q_j$ and the edges of $F \setminus Q_j$ with the color 1, the edges of $F \cap Q_j$ with the color 2, and the edges of $P_1 \setminus Q_j$ with 0. The obtained coloring ψ_j is a weak coloring, and its weakness set is comprised of the vertices incident to the edges of $P_1 \cap Q_j$.

If Q_j is not a 1-factor, the definition of ψ_j is similar although not so uniform. We keep the assignment $\psi_j(e) = 0$ for each edge $e \in P_1 - Q_j$. Recall that Q_j/F is now a spanning forest of G/F. Thus we can order the vertices of G/F as $w_1, w_2, \dots, w_m$ in such a way that each w_k is adjacent in Q_j/F to at most one of its predecessors. Give the circuits of F the corresponding ordering $C_1, C_2, \dots, C_m$.

Now color the edges of C_1 by 1 and 2 in such a way that two consecutive edges of C_1 have the same color if and only if the third edge incident with their common vertex belongs to $Q_j \cap P_1$. Furthermore, for each edge $f \in Q_j \cap P_1$ incident with C_1 define $\psi_j(f)$ to be the color of the two adjacent edges of C_1. Note that such a coloring is possible since Q_j/F is a parity subgraph of G/F.

Process the circuits of F in order. If an edge t of $Q_j \cap P_1$ is incident with a circuit C_k and with some predecessor, assign the two adjacent edges of C_k the color $\psi_j(t)$ already defined. Extend the coloring to the whole of C_k only using the colors 1 and 2 subject to the condition that two consecutive edges of C_k have the same color if and only if they are incident with an edge of $Q_j \cap P_1$. Continue by defining $\psi_j(f)$ for each edge $f \in Q_j \cap P_1$ incident with C_k to be the color of the two adjacent edges of C_k. Since there is at most one adjacency with a predecessor circuit, the result is a weak 3-edge-coloring of G with weakness set consisting of the vertices incident with an edge of $Q_j \cap P_1$.

Since both $Q_2 \cap P_1$ and $Q_3 \cap P_1$ are matchings and $(Q_2 \cap P_1) \cap (Q_3 \cap P_1) = Q_1 \cap Q_2 \cap Q_3 = \emptyset$, the weakness sets of ψ_2 and ψ_3 are disjoint and the pair (ψ_2, ψ_3) is a proper affine edge-coloring.

By combining the possibilities for ψ_2 and ψ_3 around any given vertex of G, we can verify that the coloring $\psi = (\psi_2, \psi_3)$ uses the first i of the following lines of the affine plane $AG(2,3)$:

$$\{(0,0),(1,1),(2,2)\}, \quad \{(0,0),(1,2),(2,1)\}, \quad \{(2,0),(2,1),(2,2)\},$$
$$\{(0,2),(1,2),(2,2)\}, \quad \{(0,1),(1,1),(2,1)\}, \text{ and } \{(1,0),(1,1),(1,2)\}.$$

As these first i lines form an A_i-configuration, ψ is the A_i-coloring sought. □

We finish this section with a theorem which establishes another necessary and sufficient condition for the existence of an F_5-coloring. A *cut* in G is the set of all edges that have exactly one vertex in each of X and X' for some partition $\{X, X'\}$ of $V(G)$. A cut is *odd* if either X or X' has an odd number of vertices. Observe that in a cubic graph, a cut is odd whenever it contains an odd number of edges.

Theorem 3.3. *A cubic graph G admits an F_5-coloring if and only if it contains two 1-factors M_1 and M_2 such that each odd cut in G has an edge outside $M_1 \cap M_2$.*

Proof. Assume that G has an F_5-coloring. Then, according to Theorem 3.1, it contains a triad of weight 2, that is to say, two 1-factors M_1 and M_2 and a parity subgraph P with no edge in common. Since P is a parity subgraph of G, it intersects every odd cut of G. However, $M_1 \cap M_2 \cap P = \emptyset$, so every odd cut must have an edge outside $M_1 \cap M_2$, as asserted.

For the converse, assume that G contains two 1-factors M_1 and M_2 such that each odd cut has an edge outside $M_1 \cap M_2$. Then every component of $H = G \setminus (M_1 \cap M_2)$ has even order. It is a routine matter to find a parity subgraph K of G included in H. Consequently, $M_1 \cap M_2 \cap K = \emptyset$. Thus $\{M_1, M_2, K\}$ is a triad of weight 2 in G. By Theorem 3.1, G admits an F_5-coloring. □

4 Colorings by general Steiner triple systems

We illustrate the power of Theorem 3.2 by giving a new proof of the main result of the paper by Holroyd and Škoviera [18], which states that every bridgeless cubic graph has an $\mathcal{S}$-coloring for each non-trivial Steiner triple system $\mathcal{S}$. Máčajová and Škoviera [26] established the following.

Theorem 4.1. [26] *Every bridgeless cubic graph G admits an F_6-coloring.*

Before the next theorem, we need a lemma proved by Holroyd and Škoviera [18, Section 5].

Lemma 4.2. *Every non-trivial Steiner triple system either contains a copy of F_6, a copy of D_8 or a copy of $D_9 \cong A_6$.*

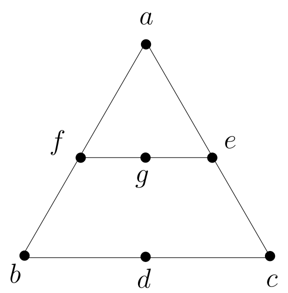

Figure 5: The C_{14}-configuration

Theorem 4.3. [18] *Every bridgeless cubic graph has an $\mathcal{S}$-coloring for every non-trivial Steiner triple system $\mathcal{S}$.*

Proof. By Theorems 4.1 and 4.1, every bridgeless cubic graph admits both an F_6-coloring and a D_9-coloring. Since D_8 is a homomorphic image of D_9 (which arises by identifying the points h and i of D_9, see Figure 1), every bridgeless cubic graph can be D_8-colored as well. By Lemma 4.2, at least one of these three configurations is contained in every non-trivial Steiner triple system $\mathcal{S}$. Thus every bridgeless cubic graph has an $\mathcal{S}$-coloring for every such $\mathcal{S}$, as stated. □

5 Abelian colorings

Given an abelian group A, an *A-coloring* of a cubic graph G is an assignment of non-zero elements of A to the edges of G in such a way that the sum of colors at each vertex equals 0. An A-coloring can be either *improper* or *proper* according to whether adjacent edges can have or must not have equal colors.

The study of abelian colorings was initiated by Archdeacon [1] in 1986 (see also [2]). In response to his paper, Máčajová et al. [27] proved that every bridgeless cubic graph has an improper A-coloring for every abelian group A of order at least 5, thereby establishing Archdeacon's conjecture. Proper abelian colorings have been first studied by Máčajová et al. [27], where it was indicated that the analogous existence problem for proper colorings is much more difficult. In this section we deal with proper A-colorings in a greater detail. Since henceforth we consider only proper A-colorings, we omit the adjective "proper". In particular, we say that a cubic graph is *A-colorable* if it admits a *proper* A-coloring.

Let A be an abelian group. Form a partial Steiner triple system $\mathcal{C}(A)$ by taking all 3-element subsets $\{x, y, z\}$ of $A - \{0\}$ with $x+y+z = 0$ as its blocks. Then a proper A-coloring is nothing but a $\mathcal{C}(A)$-coloring. This fact enables us to investigate abelian colorings by the methods developed in the previous sections.

An abelian group A is *class* 1 or *class* 2 according to whether the configuration $\mathcal{C}(A)$ is class 1 or class 2. If $|A| \leq 5$ and A is not the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$, then A is neither class 1 nor class 2, because $\mathcal{C}(A) = \emptyset$. The Klein group and the cyclic groups of order 6, 7, 8, and 9 are class 1. If $|A| \geq 6$, the configuration $\mathcal{C}(A)$ covers all non-zero elements of A; in particular $\mathcal{C}(A) \neq \emptyset$.

We summarize these facts in the following proposition whose proof is left to the reader.

Proposition 5.1. *Let A be an abelian group.*

(1) *The configuration $\mathcal{C}(A)$ is non-empty if and only if $A = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $|A| \geq 6$. Moreover, if $\mathcal{C}(A) \neq \emptyset$ then $\mathcal{C}(A)$ covers all points of $A - \{0\}$.*

(2) *If A is one of $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, and $\mathbb{Z}_9$, then a cubic graph is A-colorable if and only if it is 3-edge-colorable.*

Our next aim is to show that all sufficiently large groups are class 2.

Theorem 5.2. *If A is an abelian group of order at least 12 or $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then every bridgeless cubic graph is A-colorable.*

Proof. Let us express A as a direct product of cyclic groups—say $A = \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \cdots \times \mathbb{Z}_{k_m}$ where $k_1 \geq \cdots \geq k_m$. If $k_1 = 2$, then also $k_2 = k_3 = 2$, so A contains a subgroup B

group	a	b	c	d	e	f	g	h	i
$\mathbb{Z}_{k_1}$									
$k_1 = 12$ or $k_1 \geq 15$	1	$k_1 - 3$	4	$k_1 - 1$	$k_1 - 5$	2	3	8	$k_1 - 6$
$k_1 = 13$ or $k_1 = 14$	1	$k_1 - 3$	5	$k_1 - 2$	$k_1 - 6$	2	4	9	$k_1 - 7$
$\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2}$									
$k_1 = 4,\ k_2 = 4$	$(0,2)$	$(3,2)$	$(1,3)$	$(0,3)$	$(3,3)$	$(1,0)$	$(0,1)$	$(2,3)$	$(2,1)$
$k_1 = 5,\ k_2 = 5$	$(0,1)$	$(4,4)$	$(1,2)$	$(0,4)$	$(4,2)$	$(1,0)$	$(0,3)$	$(2,4)$	$(3,3)$
$k_1 = 6$ or $k_1 \geq 10,\ k_2 = 2$ or $k_1 \geq 6,\ k_2 \geq 3$	$(1,0)$	$(k_1 - 3, 0)$	$(1,1)$	$(2, k_2 - 1)$	$(k_1 - 2, k_2 - 1)$	$(2,0)$	$(0,1)$	$(5,1)$	$(k_1 - 3, k_2 - 1)$
$k_1 = 8,\ k_2 = 2$	$(1,0)$	$(k_1 - 1, 1)$	$(2,1)$	$(k_1 - 1, 0)$	$(1, k_2 - 3)$	$(1,0)$	$(0,3)$	$(0,4)$	$(0, k_2 - 2)$
$\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \mathbb{Z}_{k_3}$									
$k_1 = 3,\ k_2 = 3,\ k_3 = 3$	$(0,0,1)$	$(1,1,1)$	$(2,1,1)$	$(0,1,1)$	$(1,2,1)$	$(2,2,1)$	$(0,2,1)$	$(1,0,1)$	$(2,0,1)$
$k_1 = 4,\ k_2 = 2,\ k_3 = 2$	$(0,1,0)$	$(3,1,0)$	$(2,1,1)$	$(3,0,1)$	$(2,0,1)$	$(1,0,0)$	$(1,0,1)$	$(3,1,1)$	$(1,1,1)$

Table 1: D_9-configuration in some abelian groups.

isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. By Jaeger's 8-flow theorem [21], G has a nowhere-zero B-flow which is a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-coloring of G.

Now let $k_1 \geq 3$. Consider the subgroup B of A isomorphic to $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \cdots \times \mathbb{Z}_{k_r}$, where r is the smallest integer such that $|B| \geq 12$. Thus, $r \leq 3$. Taking into account that the direct product of cyclic groups of coprime orders is again cyclic, it can be deduced from Table 1 that for each such B the configuration $\mathcal{C}(B)$ contains a copy of D_9. Since $\mathcal{C}(B) \subseteq \mathcal{C}(A)$, there is a copy of D_9 in $\mathcal{C}(A)$ for every abelian group A of order at least 12. The result now follows from the fact that, by Theorem 4.1, every bridgeless cubic graph D_9-colorable. □

There are exactly four non-isomorphic abelian groups not treated by Proposition 5.1 and Theorem 5.2, namely $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_{10} \cong \mathbb{Z}_5 \times \mathbb{Z}_2$, and $\mathbb{Z}_{11}$. We call them the *exceptional groups*. Since each of them contains an F_4-configuration, we propose the following conjecture.

Conjecture 5.3. *Every bridgeless cubic graph has an A-coloring for every abelian group $A \in \{\mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_{10}, \mathbb{Z}_{11}\}$.*

It can be verified that the configurations corresponding to the exceptional groups are all non-isomorphic and neither of them can be mapped to another. Surprisingly, however, the smallest among the exceptional groups, the group $\mathbb{Z}_4 \times \mathbb{Z}_2$, plays a special role.

Theorem 5.4. *If a cubic graph is $\mathbb{Z}_4 \times \mathbb{Z}_2$-colorable, then it is A-colorable for every $A \in \{\mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_{10}, \mathbb{Z}_{11}\}$.*

Proof. If a cubic graph G has a $\mathbb{Z}_4 \times \mathbb{Z}_2$-coloring, then it also has an F_5-coloring. The isomorphism $\mathcal{C}(\mathbb{Z}_4 \times \mathbb{Z}_2) \cong F_5$ is indicated in Figure 6. By Theorem 3.2, G has an A_5-coloring as well. Since for each $A \in \{\mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_{10}, \mathbb{Z}_{11}\}$ the configuration $\mathcal{C}(A)$ contains a copy of A_5 (see Figure 7), G has an A-coloring for each $A \in \{\mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_{10}, \mathbb{Z}_{11}\}$. □

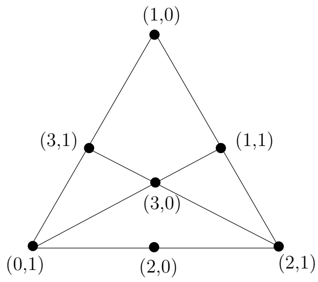

Figure 6: $\mathcal{C}(\mathbb{Z}_4 \times \mathbb{Z}_2) \cong F_5$

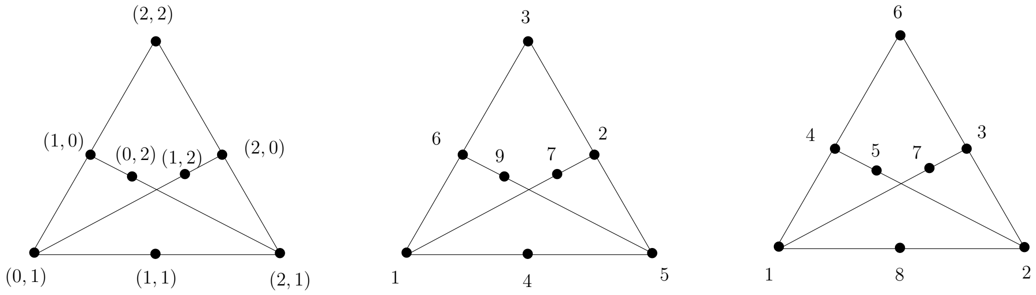

Figure 7: A_5-configuration in $\mathcal{C}(\mathbb{Z}_3 \times \mathbb{Z}_3)$, $\mathcal{C}(\mathbb{Z}_{10})$, and $\mathcal{C}(\mathbb{Z}_{11})$

6 Variations on an abelian theme

We introduce three different modifications of the concept of an abelian coloring. In the first of them we simply extend the set of available colors with the zero elements of the group. The other modifications draw their inspiration from the analogy with nowhere-zero flows.

Define an *extended A-coloring* of a cubic graph to be a proper edge-coloring by elements of A, including 0, such that the colors of any three pairwise adjacent edges sum to 0. Let $\mathcal{C}^*(A)$ be the extended configuration for A whose blocks are all three-element subsets of A. An extended A-coloring is nothing but a $\mathcal{C}^*(A)$-coloring.

The following theorem yields a similar classification of abelian groups as Proposition 5.1 and Theorem 5.2 for the case of the abelian colorings.

Theorem 6.1. *Let A be an abelian group.*

(1) *The configuration $\mathcal{C}^*(A)$ is non-empty if and only if $|A| \geq 3$.*

(2) *If A is any of $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, and $\mathbb{Z}_2 \times \mathbb{Z}_2$, then a cubic graph has an extended A-coloring if and only if it is 3-edge-colorable.*

(3) *Let A be an abelian group of order at least 8. Then every bridgeless cubic graph has an extended A-coloring.*

Proof. Claim (1) is trivial. Since for each group listed in (2) the configuration $\mathcal{C}^*(A)$ maps to the trivial configuration I, every extended A-coloring induces a 3-edge-coloring.

We now prove (3). Let A be an abelian group. If $|A| \geq 12$ or $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then the conclusion follows from Theorem 5.2. Thus we are left with groups such that $8 \leq |A| < 12$ other than $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. As in the proof of Theorem 5.2, it is sufficient to show that $\mathcal{C}^*(A)$ contains a copy of one of the configurations F_6, D_8, and D_9; this is a consequence of Theorem 4.1 and the fact that D_8 is a homomorphic image of D_9. By a direct verification one can see that $\mathcal{C}^*(\mathbb{Z}_4 \times \mathbb{Z}_2)$ contains F_6, $\mathcal{C}^*(\mathbb{Z}_8)$ and $\mathcal{C}^*(\mathbb{Z}_4 \times \mathbb{Z}_2)$ contain D_8, and $\mathcal{C}^*(\mathbb{Z}_9)$, $\mathcal{C}^*(\mathbb{Z}_3 \times \mathbb{Z}_3)$, $\mathcal{C}^*(\mathbb{Z}_{10})$, and $\mathcal{C}^*(\mathbb{Z}_{11})$ contain D_9. □

The only non-trivial abelian group not covered by Theorem 6.1 is $\mathbb{Z}_7$. The extended configuration for this group is isomorphic to the mitre-configuration introduced in Section 2. Thus extended $\mathbb{Z}_7$-colorings provide the Mitre Conjecture from Section 2 with an algebraic interpretation.

We proceed with the second variation on the definition of an abelian coloring. The relationship between flows with values in finite abelian groups of order k and integer nowhere-zero k-flows suggests the following definition. An *integer k-coloring* of a cubic graph G is a $\mathbb{Z}$-coloring σ satisfying the condition that $0 < |\sigma(e)| < k$ for each edge e of G. Let I_k be the configuration whose points are all non-zero integers n with $|n| < k$ and blocks are all three-element subsets with zero sum. Then an integer k-coloring is exactly an I_k-coloring.

As we show next, integer colorings are closely related to both Fano and abelian colorings.

Theorem 6.2. *The following two statements hold for every bridgeless cubic graph G.*

(1) *For $i = 4, 5$ and 6, if G admits an F_i-coloring, then it also admits an integer $(i+2)$-coloring.*

(2) *If G admits an integer 6-coloring, then it admits both a $\mathbb{Z}_{10}$-coloring and a $\mathbb{Z}_{11}$-coloring.*

Proof. (1) Let $i \in \{4, 5, 6\}$ and assume that G has an F_i-coloring. Theorem 3.2 implies that G also admits an A_i-coloring. As shown in Figure 8, the configuration I_{i+2} contains a copy of A_i. Hence G also admits an integer $(i+2)$-coloring.

(2) Let σ be an integer 6-coloring of G. Define $\sigma'(e)$ as the reduction of $\sigma(e)$ modulo 10 and $\sigma''(e)$ as the reduction of $\sigma(e)$ modulo 11. Since reduction modulo 11 establishes a bijection from the point-set of I_6 to $\mathbb{Z}_{11} - \{0\}$ that preserves the zero sum, we see that σ'' is a $\mathbb{Z}_{11}$-coloring. The argument for σ' is similar, except that the elements 5 and -5 collapse into the same element of $\mathbb{Z}_{10} - \{0\}$. Fortunately, in σ, the colors 5 and -5 cannot occur on adjacent edges, because otherwise the color of the third edge at their common vertex would have to be 0. Therefore σ' is a proper edge-coloring and consequently it is a $\mathbb{Z}_{10}$-coloring of G. □

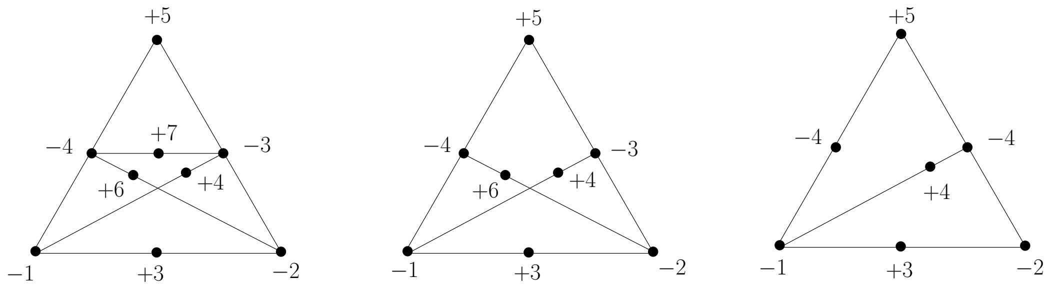

Figure 8: A_i-configuration in I_{i+2}

By combining the previous result with Theorem 4.1 we obtain the following corollary.

Corollary 6.3. *Every bridgeless cubic graph has an integer* 8*-coloring.*

Observe that I_5 is a class 1 configuration while I_6 contains a copy of F_4. This leads us to propose the following two conjectures, the latter being a weaker form of the former.

Conjecture 6.4. *Every bridgeless cubic graph admits an integer* 6*-coloring.*

Conjecture 6.5. *Every bridgeless cubic graph admits an integer* 7*-coloring.*

As our third variation we could consider *extended integer k-colorings* defined analogously as above except that 0 would become an admissible color. This definition, however, does not bring anything new: the extended configuration I_4^* is isomorphic to the mitre while I_5^* contains A_6. Thus a bridgeless cubic graph has an extended integer 4-coloring if and only if it admits an extended $\mathbb{Z}_7$-coloring. Finally, by Theorem 4.1, every bridgeless cubic graph has an extended integer 5-coloring.

7 Concluding remarks

We have presented a systematic approach to edge-colorings of cubic graphs based on configurations with 3-element blocks. There are many other configurations besides those considered in this paper. Perhaps the first type of configurations to try are so called *symmetric configurations n_3*.

In general, a *symmetric configuration n_k* consists of n points and n lines (or blocks) arranged in such a way that k lines pass through each point, and there are k points on each line. Furthermore, there is at most one line through any pair of points. With each symmetric configuration one can associate a bipartite cubic graph, called the *incidence graph*—or the *Levi graph*—of a configuration. The parts of the incidence graph correspond to the points and the lines, two vertices being adjacent if the corresponding point and the line are incident. It is well known [5, Proposition 1] that every bipartite cubic graph of girth at least six uniquely determines a symmetric configuration, and vice versa. Exchanging the roles of the parts results in the *dual configuration*.

Although many interesting symmetric configurations are of geometric origin, the terms *point* and *line* need not have any geometric significance. Symmetric configurations were defined by Reye [29] in 1876 and as such belong to the oldest combinatorial structures. For modern investigation of configurations the reader is referred to [4, 5, 16, 17, 28]. In particular, Betten et al. [4] lists all small n_3-configurations.

The smallest symmetric configuration is the Fano plane, the unique 7_3-configuration. Its incidence graph is the Heawood graph. There is a single 8_3-configuration known as the Möbius-Kantor configuration. It is isomorphic to the affine plane $AG(2,3)$ minus a point which in turn is isomorphic to $\mathcal{C}(\mathbb{Z}_3 \times \mathbb{Z}_3)$. Its incidence graph is the generalized Petersen graph $GP(8,3)$. There exist exactly three non-isomorphic 9_3-configurations: the Pappus configuration from his famous Hexagon Theorem is easily seen to be class 1. Its incidence graph is the Haar graph $H(261)$ described by Pisanski and Randić [28]. The remaining two 9_3-configurations are both class 2 and contain a copy of F_4, but no copy of F_5 or the mitre.

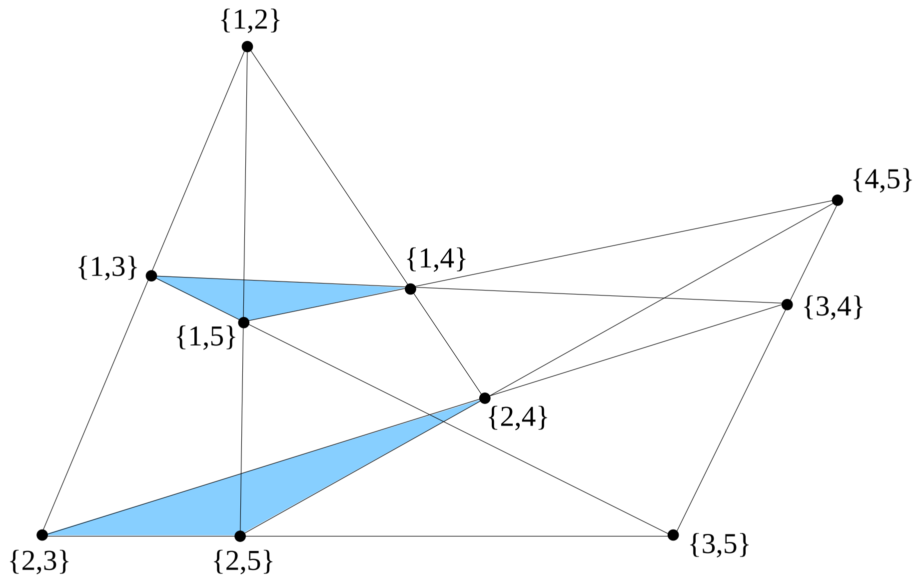

Figure 9: Desargues configuration

One of the most famous geometric configurations, the Desargues configuration shown in Figure 9, is a configuration of type 10_3. Its incidence graph is the generalized Petersen graph

$GP(10,3)$. The configuration arises in the following Theorem of Desargues from projective geometry: *If two triangles are perspective from a point, they are perspective from a line, and conversely*; see, e.g., Coxeter's textbook [8, p. 238]. Surprisingly, the same configuration arises in graph theory in connection with the Cycle Double Cover Conjecture [22]. Its more specific form, the 5-Cycle Double Cover Conjecture (5-CDC), asserts that every bridgeless graph admits a 5-*cycle double cover*, that is, a collection of five even subgraphs such that each edge belongs to exactly two of them. It is well known that the 5-CDC is equivalent to its restriction on cubic graphs, and currently it is known to be true for cubic graphs of oddness at most 4 [20].

We next observe that the 5-CDC is equivalent to the statement that the Desargues configuration colors every bridgeless cubic graph.

Theorem 7.1. *A cubic graph has a* 5*-cycle double cover if and only if it has a* D*-coloring, where* D *is the Desargues configuration.*

Proof. Assume that a cubic graph G has a double cover by five even subgraphs $H_1, \ldots, H_5$. Color each edge e of G by a two-element subset $\{j,k\} \subseteq \{1,2,\ldots,5\}$ whenever e belongs to both H_j and H_k. Since every vertex of G is incident with an even number of edges of each H_i (either zero or two), we deduce that, at every vertex, three of the five even subgraphs must meet each other. It follows that our coloring is proper and that the color pattern at each vertex consists of three two-element subsets which are contained in the same three-element subset of $\{1,2,\ldots,5\}$. In other words, every 5-cycle double cover of G induces a C-coloring with a configuration C isomorphic to the Desargues configuration depicted in Figure 9. The converse can be established simply by reversing the arguments. ☐

Another remarkable configuration is the Cremona-Richmnond configuration of type 15_3. Following Coxeter [7], it can be defined as follows. Let $1,2,\ldots,6$ be six points of the real projective 4-space in general position. Consider any two distinct points i and j, and let $\{i,j\}$ denote the intersection of the line passing through i and j with the hyperplane determined by the four other points. In this way we obtain fifteen points which lie by threes on fifteen lines, each of the lines being the common line of three hyperplanes. The fifteen points and fifteen lines form the *Cremona-Richmond configuration of type* 15_3. The stellar representation of this configuration given in Figure 10 is due to Boben et al. [5]. The incidence graph is the well known Tutte 8-cage (see [4, 28]).

The origins of the Cremona-Richmond configuration are quite vague. In algebraic geometry, it emerged in the studies of families of straight lines on cubic surfaces which were popular in the second half of the nineteenth century. Cremona [9] seems to have been the first to give a description which can be interpreted as the list of points and lines of this

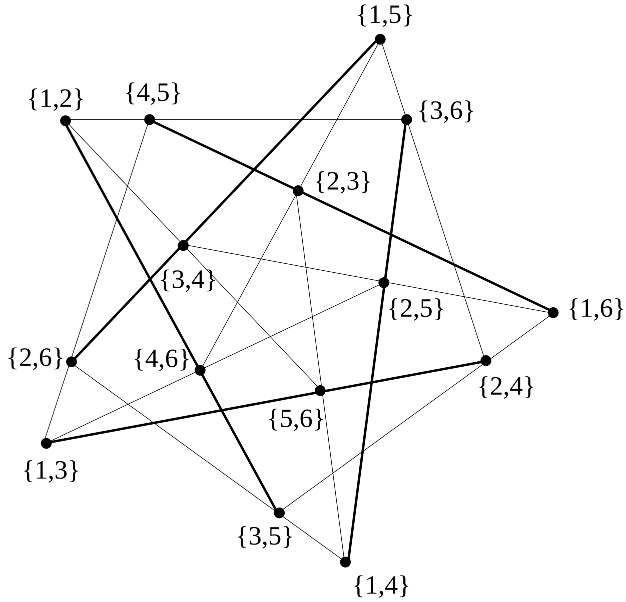

Figure 10: Cremona-Richmond configuration

configuration. Richmond [30] found its realization by points and lines in the 4-dimensional projective space over an infinite field.

We now show that the same configuration arises in connection with the famous Fulkerson's Conjecture whose origin is in mathematical programming [13]. The conjecture states that in every bridgeless cubic graph there exists a collection of six perfect matchings such that each edge belongs to exactly two of them. Such a collection is called a *double cover* by six perfect matchings.

Theorem 7.2. *A cubic graph has a double cover by six perfect matchings if and only if it has a CR-coloring where CR is the Cremona-Richmond configuration.*

Proof. Assume that a cubic graph G has a double cover by six 1-factors $M_1, \dots, M_6$, and color every edge e of G by a two-element subset $\{j,k\} \subseteq \{1,2,\dots,6\}$ whenever e belongs to both M_j and M_k. In this way, every double cover of G by six 1-factors induces a C-coloring where C is a configuration whose points are all two-element subsets of $\{1,2,\dots,6\}$ and three points form a block if and only if their union is the whole $\{1,2,\dots,6\}$. It is immediate that C is a 15_3-configuration. The fact that C is isomorphic to the Cremona-Richmond configuration follows from the labeling displayed in Figure 10. Again, the converse implication can be established by reversing the arguments. □

There is another conjecture in graph theory where the Cremona-Richmond configuration plays an important role, namely the Petersen Coloring Conjecture [23]. The conjecture states that the edges of every bridgeless cubic graph G can be mapped into the edges of the Petersen graph in such a way that any three mutually incident edges of G are mapped to three mutually incident edges of the Petersen graph. Such a mapping is called a *Petersen coloring* of G. Let us define a partial Steiner triple system P by taking its points to be the edges of the Petersen graph and its blocks to be the triples of pairwise adjacent edges. Note that a Petersen coloring of a graph is nothing but a P-coloring. The configuration P has 15 points and 10 lines. In particular it is not a symmetric configuration. However, it is a routine matter to verify that P results from the Cremona-Richmond configuration by removing a parallel class of blocks, i.e., a set of disjoint blocks covering every point. An example of such a set is indicated in Figure 10 by bold lines. We call P the *depleted* Cremona-Richmond configuration.

Theorem 7.3. *A cubic graph has a Petersen coloring if and only if it has a P-coloring, where P is the depleted Cremona-Richmond configuration.*

To conclude this section we summarize the conjectures presented in this paper and the relations between them. In Figure 11, each box represents a conjecture or a theorem. A box with a bold frame represents a theorem, otherwise it represents a conjecture. Each conjecture or theorem is encoded either by its name or by the corresponding partial Steiner triple system. A box containing the symbol of a configuration or of a group C represents the statement that *every bridgeless cubic graph is C-colorable.* An arrow between boxes means that the validity of the "initial" statement implies the validity of the "terminal" statement.

Note that the Petersen graph admits both a 5-cycle double cover and a double cover by six 1-factors. Thus if a graph G admits a Petersen coloring, then both a 5-cycle double cover and a double cover by six 1-factors of G can be obtained by "lifting" the corresponding structure from the Petersen graph to G. This explains the two implications at the bottom of Figure 11. The first of them also follows from the fact that the depleted Cremona-Richmond configuration is contained in the Cremona-Richmond configuration.

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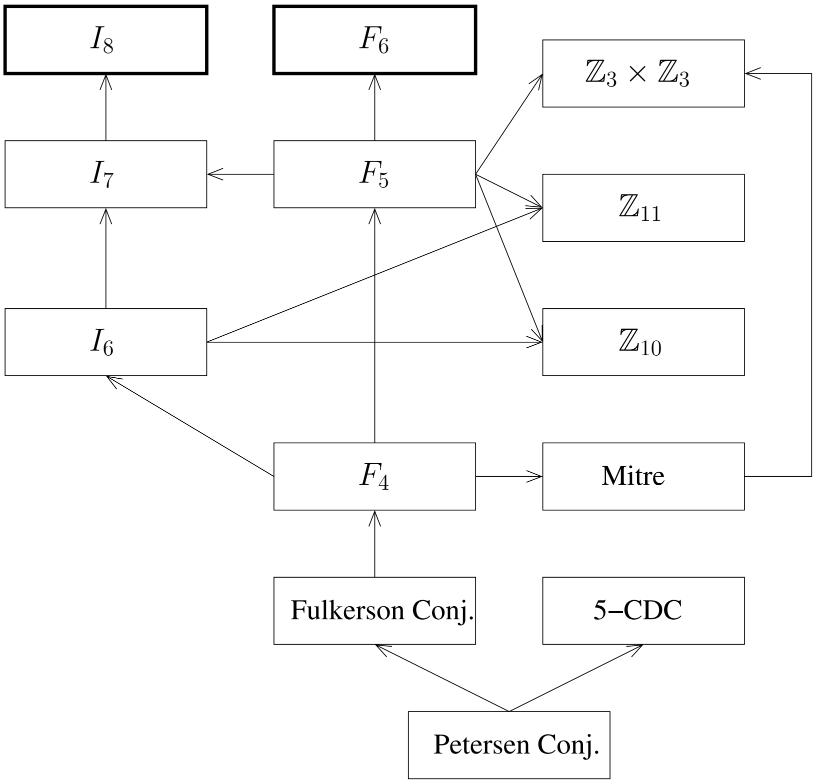

Figure 11: Relations between the conjectures

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