Families of Subsets with a Forbidden Subposet

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Abstract

Given a finite poset P, we consider the largest size $\mathrm{La}(n, P)$ of a family of subsets of $[n] := \{1, \ldots, n\}$ that contains no (weak) subposet P. Letting P_k denote the k-element chain (path poset), Sperner's Theorem (1928) gives that the largest size of an antichain of subsets of $[n]$, $\mathrm{La}(n, P_2) = \binom{n}{\lfloor n/2 \rfloor}$, and Erdős (1945) showed more generally that $\mathrm{La}(n, P_k)$ is the sum of the k middle binomial coefficients in n.

In recent years Katona and his collaborators investigated $\mathrm{La}(n, P)$ for other posets P. It can be very challenging, even for small posets. Based on results we have, Griggs and Linyuan Lu conjecture that $\pi(P) := \lim_{n\to\infty} \mathrm{La}(n, P)/\binom{n}{\lfloor n/2 \rfloor}$ exists for general posets P, and, moreover, it is an integer obtained in a specific way.

For $k \geq 2$ let D_k denote the k-diamond poset $\{A < B_1, \ldots, B_k < C\}$. Using probabilistic reasoning to bound the average number of times a random full chain meets a P-free family $\mathcal{F}$, called the Lubell function of $\mathcal{F}$, Griggs, Wei-Tian Li, and Lu prove that $\pi(D_2) < 2.273$, if it exists. This is a stubborn open problem, since we expect $\pi(D_2) = 2$. It is then surprising that, with appropriate partitions of the set of full chains, we can explicitly determine $\pi(D_k)$ for infinitely many values of k, and, moreover, describe the extremal D_k-free families. For these fortunate values of k, and for a growing collection of other posets P, we have that $\mathrm{La}(n, P)$ is a sum of middle binomial coefficients in n, while for other values of k and for most P, it seems that $\mathrm{La}(n, P)$ is far more complicated.

Some techniques being used are adapted from Turán theory of graphs and hypergraphs, including probabilistic arguments and, more recently, flag algebras.

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