Graph Sparsity

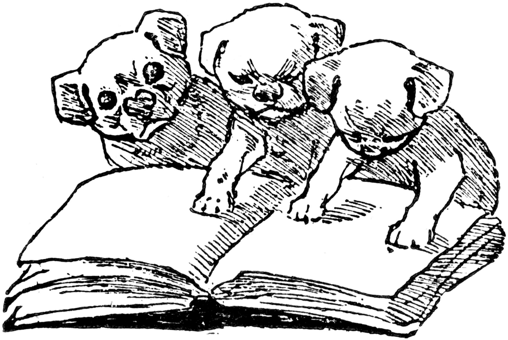
Leçons sur les graphes épars

Patrice Ossona de Mendez, Jaroslav Nešetřil

CAMS, CNRS/EHESS, Paris, France — Charles University, Praha, Czech Republic

LIAFA, 6–8 October 2010

Introduction & Prolegomena

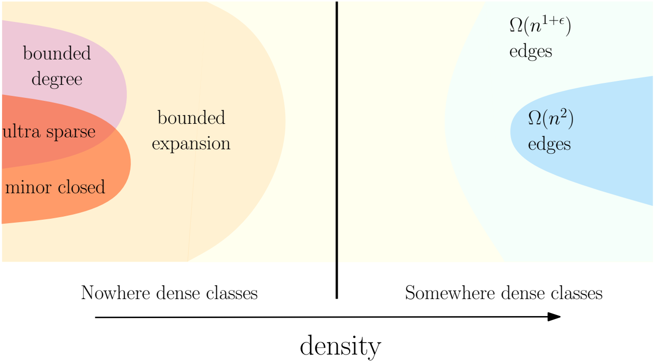


Introduction

1. Introduction
2. Density related parameters
3. Minors
4. Ordering and orientation related parameters
5. Homomorphism related parameters
6. Ramsey theory and extremal graph theory
7. Shallow subdivisions
8. Girth

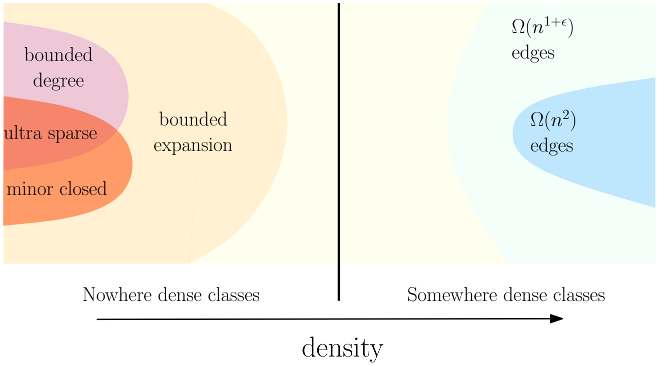
What is a sparse graph?

- No answer for a single graph $\rightarrow$ graph sequence, graph class.
- Existence of qualitative jumps? thresholds? Uniqueness?
- Based on density? subgraph counting? decomposition properties? FO-properties?



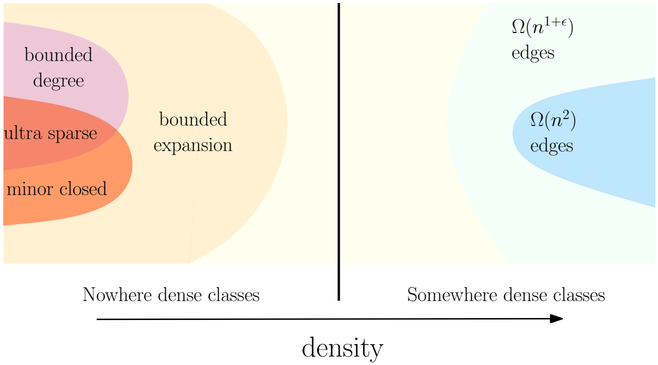
What is a sparse graph?

- No answer for a single graph → graph sequence, graph class.
- Existence of qualitative jumps? thresholds? Uniqueness?
- Based on density? subgraph counting? decomposition properties? FO-properties?

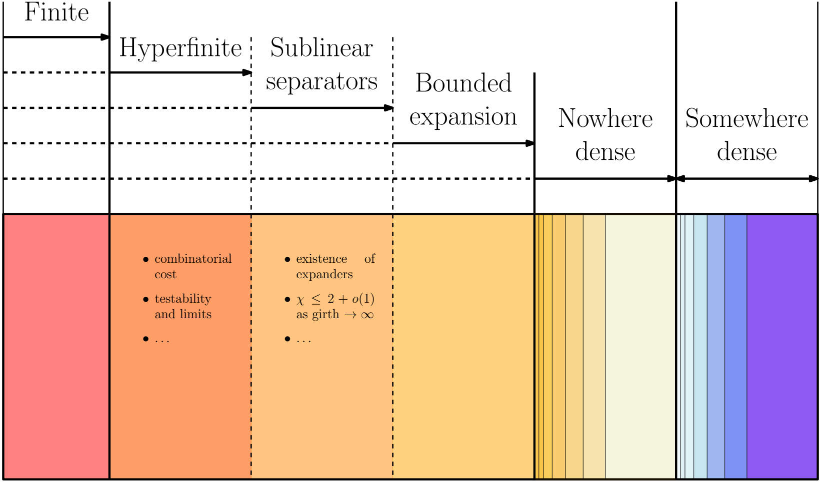


What is a sparse graph?

- No answer for a single graph → graph sequence, graph class.
- Existence of qualitative jumps? thresholds? Uniqueness?
- Based on density? subgraph counting? decomposition properties? FO-properties?



Thresholds



Finite
Hyperfinite
Sublinear separators
Bounded expansion
Nowhere dense
Somewhere dense
• combinatorial cost
• testability and limits
• ...
• existence of expanders
• $\chi \leq 2 + o(1)$ as girth $\to \infty$
• ...

Density related parameters

1 Introduction

2 Density related parameters

3 Minors

4 Ordering and orientation related parameters

5 Homomorphism related parameters

6 Ramsey theory and extremal graph theory

7 Shallow subdivisions

8 Girth

Average degree and minimum degree

Every graph G has a non-empty induced subgraph H such that

$$\delta(H) \geq \frac{\|H\|}{|H|} \geq \frac{\|G\|}{|G|}.$$

For every $\epsilon > 1/|G|$, there exists a non-empty induced subgraph H_ϵ of G such that

$$\delta(H_\epsilon) \geq (1-\epsilon)\frac{\|G\|}{|G|} \qquad \text{and} \qquad \|H_\epsilon\| \geq \epsilon\|G\|.$$

Average degree and minimum degree

Every graph G has a non-empty induced subgraph H such that

$$\delta(H) \geq \frac{\|H\|}{|H|} \geq \frac{\|G\|}{|G|}.$$

For every $\epsilon > 1/|G|$, there exists a non-empty induced subgraph H_ϵ of G such that

$$\delta(H_\epsilon) \geq (1-\epsilon)\frac{\|G\|}{|G|} \qquad \text{and} \qquad \|H_\epsilon\| \geq \epsilon\|G\|.$$

Degeneracy

A graph G is k-degenerate if each nonempty subgraph of G contains a vertex of degree at most k. The maximum average degree of G, denoted $\mathrm{mad}(G)$, is the maximum average degrees of the subgraphs of G:

$$\mathrm{mad}(G) = \max_{H \subseteq G} \overline{\mathrm{d}}(H) = \max_{H \subseteq G} \frac{2\|H\|}{|H|}$$

Thus:

$$k \geq \lfloor \mathrm{mad}(G) \rfloor \quad \Rightarrow \quad G \text{ is } k\text{-degenerate} \quad \Rightarrow \quad \mathrm{mad}(G) < 2k.$$

Degeneracy

A graph G is k-degenerate if each nonempty subgraph of G contains a vertex of degree at most k. The maximum average degree of G, denoted $\mathrm{mad}(G)$, is the maximum average degrees of the subgraphs of G:

$$\mathrm{mad}(G) = \max_{H \subseteq G} \overline{\mathrm{d}}(H) = \max_{H \subseteq G} \frac{2\|H\|}{|H|}$$

Thus:

$$k \geq \lfloor \mathrm{mad}(G) \rfloor \quad \Rightarrow \quad G \text{ is } k\text{-degenerate} \quad \Rightarrow \quad \mathrm{mad}(G) < 2k.$$

Density and other parameters

$$\forall A \subseteq V, \quad \|G[A]\| \leq d|A| \qquad \text{(mad, degeneracy)}$$

$$\forall A \subseteq V, \quad \|G[A]\| \leq k(|A|-1) \qquad \text{(arboricity)}$$

$$\forall A \subseteq V, \quad \|G[A]\| \leq 2|A|-3 \qquad \text{(3T2, contacts of segments)}$$

$$\forall A \subseteq V, \quad \|G[A]\| \leq 3|A|-6 \qquad \text{(Planarity, Euler)}$$

$$\forall A \subseteq V, \quad \|G[A]\| \leq (k+\delta)(|A|-1) \qquad \text{(9 Dragons Tree Problem)}$$

Connection with minors.

Minors

1 Introduction

2 Density related parameters

3 Minors

4 Ordering and orientation related parameters

5 Homomorphism related parameters

6 Ramsey theory and extremal graph theory

7 Shallow subdivisions

8 Girth

Minors

- $G \leq_m H$: minor relation
  - well quasi order (Robertson, Seymour)
  - Hadwiger's conjecture (proved for $k \leq 6$ Robertson, Seymour, and Thomas 1993; true for almost all graphs)
- $G \leq_t H$: topological minor relation
  - not a well quasi order
  - Hajós' conjecture (false for almost all graphs, but true if large girth)
- $G \leq_i H$: immersion relation
  - well quasi order (Robertson, Seymour; 2010)
  - conjecture of Abu-Khzam and Langston (proved for $k \leq 7$, DeVos, Kawarabayashi, Mohar, and Okamura 2009)

Minors

- $G \leq_m H$: minor relation
  - well quasi order (Robertson, Seymour)
  - Hadwiger's conjecture (proved for $k \leq 6$ Robertson, Seymour, and Thomas 1993; true for almost all graphs)
- $G \leq_t H$: topological minor relation
  - not a well quasi order
  - Hajós' conjecture (false for almost all graphs, but true if large girth)
- $G \leq_i H$: immersion relation
  - well quasi order (Robertson, Seymour; 2010)
  - conjecture of Abu-Khzam and Langston (proved for $k \leq 7$, DeVos, Kawarabayashi, Mohar, and Okamura 2009)

Minors

- $G \leq_m H$: minor relation
  - well quasi order (Robertson, Seymour)
  - Hadwiger's conjecture (proved for $k \leq 6$ Robertson, Seymour, and Thomas 1993; true for almost all graphs)
- $G \leq_t H$: topological minor relation
  - not a well quasi order
  - Hajós' conjecture (false for almost all graphs, but true if large girth)
- $G \leq_i H$: immersion relation
  - well quasi order (Robertson, Seymour; 2010)
  - conjecture of Abu-Khzam and Langston (proved for $k \leq 7$, DeVos, Kawarabayashi, Mohar, and Okamura 2009)

Minors

- $G \leq_m H$: minor relation
  - well quasi order (Robertson, Seymour)
  - Hadwiger's conjecture (proved for $k \leq 6$ Robertson, Seymour, and Thomas 1993; true for almost all graphs)
- $G \leq_t H$: topological minor relation
  - not a well quasi order
  - Hajós' conjecture (false for almost all graphs, but true if large girth)
- $G \leq_i H$: immersion relation
  - well quasi order (Robertson, Seymour; 2010)
  - conjecture of Abu-Khzam and Langston (proved for $k \leq 7$, DeVos, Kawarabayashi, Mohar, and Okamura 2009)

Minors

- $G \leq_m H$: minor relation
  - well quasi order (Robertson, Seymour)
  - Hadwiger's conjecture (proved for $k \leq 6$ Robertson, Seymour, and Thomas 1993; true for almost all graphs)
- $G \leq_t H$: topological minor relation
  - not a well quasi order
  - Hajós' conjecture (false for almost all graphs, but true if large girth)
- $G \leq_i H$: immersion relation
  - well quasi order (Robertson, Seymour; 2010)
  - conjecture of Abu-Khzam and Langston (proved for $k \leq 7$, DeVos, Kawarabayashi, Mohar, and Okamura 2009)

Minors

- $G \leq_m H$: minor relation
  - well quasi order (Robertson, Seymour)
  - Hadwiger's conjecture (proved for $k \leq 6$ Robertson, Seymour, and Thomas 1993; true for almost all graphs)
- $G \leq_t H$: topological minor relation
  - not a well quasi order
  - Hajós' conjecture (false for almost all graphs, but true if large girth)
- $G \leq_i H$: immersion relation
  - well quasi order (Robertson, Seymour; 2010)
  - conjecture of Abu-Khzam and Langston (proved for $k \leq 7$, DeVos, Kawarabayashi, Mohar, and Okamura 2009)

Minors

- $G \leq_m H$: minor relation
  - well quasi order (Robertson, Seymour)
  - Hadwiger's conjecture (proved for $k \leq 6$ Robertson, Seymour, and Thomas 1993; true for almost all graphs)
- $G \leq_t H$: topological minor relation
  - not a well quasi order
  - Hajós' conjecture (false for almost all graphs, but true if large girth)
- $G \leq_i H$: immersion relation
  - well quasi order (Robertson, Seymour; 2010)
  - conjecture of Abu-Khzam and Langston (proved for $k \leq 7$, DeVos, Kawarabayashi, Mohar, and Okamura 2009)

Minors

- $G \leq_m H$: minor relation
  - well quasi order (Robertson, Seymour)
  - Hadwiger's conjecture (proved for $k \leq 6$ Robertson, Seymour, and Thomas 1993; true for almost all graphs)
- $G \leq_t H$: topological minor relation
  - not a well quasi order
  - Hajós' conjecture (false for almost all graphs, but true if large girth)
- $G \leq_i H$: immersion relation
  - well quasi order (Robertson, Seymour; 2010)
  - conjecture of Abu-Khzam and Langston (proved for $k \leq 7$, DeVos, Kawarabayashi, Mohar, and Okamura 2009)

Minors

- $G \leq_m H$: minor relation
  - well quasi order (Robertson, Seymour)
  - Hadwiger's conjecture (proved for $k \leq 6$ Robertson, Seymour, and Thomas 1993; true for almost all graphs)
- $G \leq_t H$: topological minor relation
  - not a well quasi order
  - Hajós' conjecture (false for almost all graphs, but true if large girth)
- $G \leq_i H$: immersion relation
  - well quasi order (Robertson, Seymour; 2010)
  - conjecture of Abu-Khzam and Langston (proved for $k \leq 7$, DeVos, Kawarabayashi, Mohar, and Okamura 2009)

Minors and minimum degree

Theorem (Komlós and Szemerédi, Bollobás and Thomason)

There exists a constant c such that every graph G with minimum degree at least ck^2 satisfies $\mathrm{h}_t(G) \geq k$.

Theorem (Kostochka, Thomason)

There exists a constant $\gamma \approx 0.319$ *such that every graph G with minimum degree at least* $\gamma k \sqrt{\log(k)}$ *satisfies* $\mathrm{h}(G) \geq k$.

Theorem (Norine, Thomas)

$\forall t\ \exists N_t$ *if G is $(t-2)$-connected,* $|G| > N_t$ *and* $K_t \not\leq_m G$, *then* $\exists X \subseteq V(G)$, $|X| \leq t-5$ *s.t.* $G - X$ *is planar.*

$\implies \quad \|G\| \leq (t-2)|G| - \binom{t-1}{2}$.

Minors and minimum degree

Theorem (Komlós and Szemerédi, Bollobás and Thomason)

There exists a constant c such that every graph G with minimum degree at least ck^2 satisfies $\mathrm{h}_t(G) \geq k$.

Theorem (Kostochka, Thomason)

There exists a constant $\gamma \approx 0.319$ such that every graph G with minimum degree at least $\gamma k\sqrt{\log(k)}$ satisfies $\mathrm{h}(G) \geq k$.

Theorem (Norine, Thomas)

$\forall t\ \exists N_t$ *if* G *is* $(t-2)$*-connected,* $|G| > N_t$ *and* $K_t \not\leq_m G$*, then* $\exists X \subseteq V(G)$, $|X| \leq t-5$ *s.t.* $G - X$ *is planar.*

$\Longrightarrow \qquad \|G\| \leq (t-2)|G| - \binom{t-1}{2}$.

Minors and minimum degree

Theorem (Komlós and Szemerédi, Bollobás and Thomason)

There exists a constant c such that every graph G with minimum degree at least ck^2 satisfies $\mathrm{h}_t(G) \geq k$.

Theorem (Kostochka, Thomason)

There exists a constant $\gamma \approx 0.319$ such that every graph G with minimum degree at least $\gamma k\sqrt{\log(k)}$ satisfies $\mathrm{h}(G) \geq k$.

Theorem (Norine, Thomas)

$\forall t\ \exists N_t$ *if* G *is* $(t-2)$*-connected,* $|G| > N_t$ *and* $K_t \not\leq_m G$*, then* $\exists X \subseteq V(G)$, $|X| \leq t-5$ *s.t.* $G - X$ *is planar.*

$\Longrightarrow \qquad \|G\| \leq (t-2)|G| - \binom{t-1}{2}$.

Immersions and minimum degree

Theorem (DeVos, Mohar,... ?)

Every simple graph of minimum degree Ck contains an immersion of K_k.

Conjecture (DeVos, Kawarabayashi, Mohar, and Okamura; 2009

Every simple graph of minimum degree $k - 1$ contains an immersion of K_k.

Remark

- implies the conjecture of Abu-Khzam and Langston;
- proved for $k \leq 7$.

Immersions and minimum degree

Theorem (DeVos, Mohar,... ?)

Every simple graph of minimum degree Ck contains an immersion of K_k.

Conjecture (DeVos, Kawarabayashi, Mohar, and Okamura; 2009

Every simple graph of minimum degree $k-1$ contains an immersion of K_k.

Remark

- implies the conjecture of Abu-Khzam and Langston;
- proved for $k \leq 7$.

Immersions and minimum degree

Theorem (DeVos, Mohar,. . . ?)

Every simple graph of minimum degree Ck contains an immersion of K_k.

Conjecture (DeVos, Kawarabayashi, Mohar, and Okamura; 2009

Every simple graph of minimum degree $k - 1$ contains an immersion of K_k.

Remark

- implies the conjecture of Abu-Khzam and Langston;
- proved for $k \leq 7$.

Ordering and orientation related parameters

1. Introduction
2. Density related parameters
3. Minors
4. Ordering and orientation related parameters
5. Homomorphism related parameters
6. Ramsey theory and extremal graph theory
7. Shallow subdivisions
8. Girth

Orientation

Lemma

A graph G has an acyclic orientation $\vec{G}$ such that $\Delta^-(\vec{G}) \leq k$ if and only if G is k-degenerate.

Lemma

Let G be a graph and let $\lambda : V(G) \to \mathbb{N}$. There exists an orientation of G such that every vertex v satisfies $\mathrm{d}^-(v) \leq \lambda(v)$ if and only if

$$\forall A \subseteq V(G), \quad \|G[A]\| \leq \sum_{v \in A} \lambda(v).$$

Moreover, if $\|G\| = \sum_{v \in V(G)} \lambda(v)$ there exists an orientation of G such that $\mathrm{d}^-(v) = \lambda(v)$.

Orientation

Lemma

A graph G has an acyclic orientation $\vec{G}$ such that $\Delta^-(\vec{G}) \leq k$ if and only if G is k-degenerate.

Lemma

Let G be a graph and let $\lambda : V(G) \to \mathbb{N}$. There exists an orientation of G such that every vertex v satisfies $\mathrm{d}^-(v) \leq \lambda(v)$ if and only if

$$\forall A \subseteq V(G), \quad \|G[A]\| \leq \sum_{v \in A} \lambda(v).$$

Moreover, if $\|G\| = \sum_{v \in V(G)} \lambda(v)$ there exists an orientation of G such that $\mathrm{d}^-(v) = \lambda(v)$.

Ordering related parameters

Consider the set of all linear orderings L of $V(G)$. Define a directed graph with $(x, y) \in E$ if $x <_L y$ and $\exists$ xy-path P of length at most k s.t.

$\forall z \in P$ (internal) $x <_L z$	$\longrightarrow$	$\overrightarrow{\mathrm{WAcc}}_k(G)$
$\forall z \in P$ (internal) $y <_L z$	$\longrightarrow$	$\overrightarrow{\mathrm{Acc}}_k(G)$
P is increasing with respect to L	$\longrightarrow$	$\overrightarrow{\mathrm{Trans}}_k(G)$

Generalized coloring numbers (Kierstead and Yang):

$$\mathrm{wcol}_k(G) = 1 + \min_L \ \Delta^-(\overrightarrow{\mathrm{WAcc}}_k(G))$$

$$\mathrm{col}_k(G) = 1 + \min_L \ \Delta^-(\overrightarrow{\mathrm{Acc}}_k(G)).$$

Also, $\max_L \ \chi(\overrightarrow{\mathrm{Trans}}_k(G))$, $\max_L \ \omega(\overrightarrow{\mathrm{Trans}}_k(G))$.

Ordering related parameters

Consider the set of all linear orderings L of $V(G)$. Define a directed graph with $(x, y) \in E$ if $x <_L y$ and $\exists$ xy-path P of length at most k s.t.

$$\begin{array}{ccc} \forall z \in P \text{ (internal) } x <_L z & \longrightarrow & \overrightarrow{\mathrm{WAcc}}_k(G) \\ \forall z \in P \text{ (internal) } y <_L z & \longrightarrow & \overrightarrow{\mathrm{Acc}}_k(G) \\ P \text{ is increasing with respect to } L & \longrightarrow & \overrightarrow{\mathrm{Trans}}_k(G) \end{array}$$

Generalized coloring numbers (Kierstead and Yang):

$$\mathrm{wcol}_k(G) = 1 + \min_L \ \Delta^-(\overrightarrow{\mathrm{WAcc}}_k(G))$$

$$\mathrm{col}_k(G) = 1 + \min_L \ \Delta^-(\overrightarrow{\mathrm{Acc}}_k(G)).$$

Also, $\max_L \ \chi(\overrightarrow{\mathrm{Trans}}_k(G))$, $\quad \max_L \ \omega(\overrightarrow{\mathrm{Trans}}_k(G))$.

Ordering related parameters

Consider the set of all linear orderings L of $V(G)$. Define a directed graph with $(x, y) \in E$ if $x <_L y$ and $\exists$ xy-path P of length at most k s.t.

$$\begin{array}{ccc} \forall z \in P \text{ (internal) } x <_L z & \longrightarrow & \overrightarrow{\mathrm{WAcc}}_k(G) \\ \forall z \in P \text{ (internal) } y <_L z & \longrightarrow & \overrightarrow{\mathrm{Acc}}_k(G) \\ P \text{ is increasing with respect to } L & \longrightarrow & \overrightarrow{\mathrm{Trans}}_k(G) \end{array}$$

Generalized coloring numbers (Kierstead and Yang):

$$\mathrm{wcol}_k(G) = 1 + \min_L \ \Delta^-(\overrightarrow{\mathrm{WAcc}}_k(G))$$

$$\mathrm{col}_k(G) = 1 + \min_L \ \Delta^-(\overrightarrow{\mathrm{Acc}}_k(G)).$$

Also, $\max_L \ \chi(\overrightarrow{\mathrm{Trans}}_k(G))$, $\quad \max_L \ \omega(\overrightarrow{\mathrm{Trans}}_k(G))$.

Orientation and Counting

Lemma

Let G, H be graphs. If G is k-degenerate then it includes at most

$$\frac{1}{|\mathrm{Aut}(H)|}\sum_{t=1}^{\alpha(H)} \mathrm{Acyc}_t(H)\, k^{|H|-t}\,|G|^t$$

copies of H, where $|\mathrm{Aut}(H)|$ is the number of automorphisms of H, $\mathrm{Acyc}_t(H)$ is the number of acyclic orientations of H with t sinks, and $\alpha(H)$ is the independence number of H.

Counting Subdivisions of H

Under what conditions is it true that the number of "$\leq d$-subdivision of H" in G is bounded by $c\,|G|^{\alpha(H)}$?

Orientation and Counting

Lemma

Let G, H be graphs. If G is k-degenerate then it includes at most

$$\frac{1}{|\mathrm{Aut}(H)|}\sum_{t=1}^{\alpha(H)} \mathrm{Acyc}_t(H)\, k^{|H|-t}\, |G|^t$$

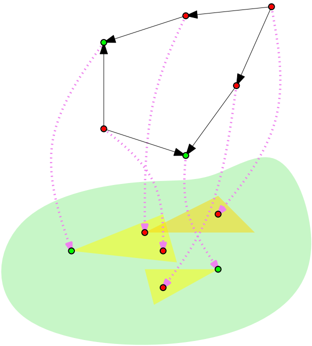
copies of H, where $|\mathrm{Aut}(H)|$ is the number of automorphisms of H, $\mathrm{Acyc}_t(H)$ is the number of acyclic orientations of H with t sinks, and $\alpha(H)$ is the independence number of H.

Counting Subdivisions of H

Under what conditions is it true that the number of "$\leq d$-subdivision of H" in G is bounded by $c\,|G|^{\alpha(H)}$?

Orientation and Counting

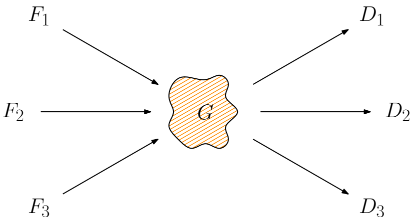
$$(\# H \subseteq G) \leq \frac{1}{|\mathrm{Aut}(H)|} \sum_{t=1}^{\alpha(H)} \mathrm{Acyc}_t(H)\, k^{|H|-t}\, |G|^t$$



Homomorphism related parameters

1. Introduction
2. Density related parameters
3. Minors
4. Ordering and orientation related parameters
5. Homomorphism related parameters
6. Ramsey theory and extremal graph theory
7. Shallow subdivisions
8. Girth

Homomorphism related parameters



clique number
independence number
$\hom(F, \cdot)$

chromatic number
decompositions
FO-definable colorings

Homomorphisms

A homomorphism $f : G \to H$ is an edge-preserving mapping $V(G) \to V(H)$. Category with product $G \times H$ and coproduct $G + H$:

$$\begin{tikzcd}
 & F \arrow[ddl, "f_1"'] \arrow[d, dashed, "f"] \arrow[ddr, "f_2"] & \\
 & G \times H \arrow[dl, "\pi_G"] \arrow[dr, "\pi_H"'] & \\
G \arrow[dr, "\iota_G"] \arrow[ddr, "g_1"'] & & H \arrow[dl, "\iota_H"'] \arrow[ddl, "g_2"] \\
 & G + H \arrow[d, dashed, "g"] & \\
 & D &
\end{tikzcd}$$

Homomorphism equivalence and core

Lemma

For any graph G there is up to isomorphism a unique graph G' which is homomorphically equivalent to G and which has the minimal number of vertices. Such a graph G' is called the core of G, and it is isomorphic to an induced subgraph of G.

$(\mathcal{G}raph / \rightleftarrows, \to)$: homomorphism order

Theorem (Density of homomorphism order)

For every pair of graphs G_1, G_2 such that $G_1 \underset{\not\leftarrow}{\rightarrow} G_2$ and $G_2 \not\to K_2$ there exists a graph G such that

$$G_1 \underset{\not\leftarrow}{\rightarrow} G \underset{\not\leftarrow}{\rightarrow} G_2.$$

Homomorphism equivalence and core

Lemma

For any graph G there is up to isomorphism a unique graph G' which is homomorphically equivalent to G and which has the minimal number of vertices. Such a graph G' is called the core of G, and it is isomorphic to an induced subgraph of G.

$(\mathcal{G}raph \,/ \rightleftarrows, \to)$: homomorphism order

Theorem (Density of homomorphism order)

For every pair of graphs G_1, G_2 such that $G_1 \mathrel{\substack{\rightarrow \\ \not\leftarrow}} G_2$ and $G_2 \not\to K_2$ there exists a graph G such that

$$G_1 \mathrel{\substack{\rightarrow \\ \not\leftarrow}} G \mathrel{\substack{\rightarrow \\ \not\leftarrow}} G_2.$$

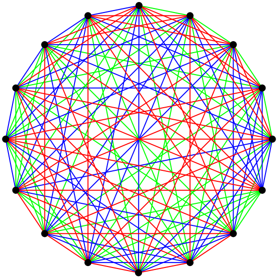
Ramsey theory and extremal graph theory

1 Introduction
2 Density related parameters
3 Minors
4 Ordering and orientation related parameters
5 Homomorphism related parameters
6 Ramsey theory and extremal graph theory
7 Shallow subdivisions
8 Girth

Ramsey theory

Theorem (Ramsey)

$\forall n_1, \ldots, n_k$ there exists $R = R(n_1, \ldots, n_k)$ (Ramsey number) which is minimum such that for every set X of cardinality at least R and every coloring of the set $\binom{X}{2}$ by k colors there exists i, $1 \le i \le k$, and a subset $Y \subseteq X$ such that $|Y| \ge n_i$ and $\binom{Y}{2}$ is monochromatic of color i.



Extremal graph theory

Theorem (Erdős, Simonovits, Stone)

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + o(n^2).$$

Theorem (Bondy, Simonovits, 1974)

$$\mathrm{ex}(n, C_{2k}) = O(n^{1+\frac{1}{k}}).$$

Extremal graph theory

Theorem (Erdős, Simonovits, Stone)

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + o(n^2).$$

Theorem (Bondy, Simonovits, 1974)

$$\mathrm{ex}(n, C_{2k}) = O(n^{1+\frac{1}{k}}).$$

Extremal graph theory

Theorem (Faudree, Simonovits, 1983)

Let $C_{m,k}$ denote the graph obtained by joining two vertices m internally disjoint paths of length k. Then

$$\mathrm{ex}(n, C_{m,k}) = O(n^{1+\frac{1}{k}}).$$

Theorem (Alon, Krivelevich and Sudakov)

Let H be a bipartite graph with maximum degree r on one side. Then there exists c_H (depending on H) such that

$$\mathrm{ex}(n, H) \leq c_H n^{2-\frac{1}{r}}.$$

Extremal graph theory

Theorem (Faudree, Simonovits, 1983)

Let $C_{m,k}$ denote the graph obtained by joining two vertices m internally disjoint paths of length k. Then

$$\mathrm{ex}(n, C_{m,k}) = O(n^{1+\frac{1}{k}}).$$

Theorem (Alon, Krivelevich and Sudakov)

Let H be a bipartite graph with maximum degree r on one side. Then there exists c_H (depending on H) such that

$$\mathrm{ex}(n, H) \leq c_H n^{2-\frac{1}{r}}.$$

Shallow subdivisions

1 Introduction

2 Density related parameters

3 Minors

4 Ordering and orientation related parameters

5 Homomorphism related parameters

6 Ramsey theory and extremal graph theory

7 Shallow subdivisions

8 Girth

Density and clique number of shallow subdivisions

Theorem (Kostochka, Pyber 1988)

Let $0 < \epsilon < 1$ and $t \in \mathbb{N}$. Let $p = \lfloor 1 + (4/\epsilon)(1 + 2\log t) \rfloor$. Any graph G with $\|G\| \geq 2^{2t(t-1)} t \cdot |G|^{1+\epsilon}$ contains a $\leq p$-subdivision of K_t.

Theorem (Jiang, 2009)

Let $0 < \epsilon < 1$ and $t \in \mathbb{N}$. Then there exists $N = N(\epsilon, t)$ such that any graph G of order at least N with $\|G\| \geq 2^{7t^2} \cdot |G|^{1+\epsilon}$ contains a $\leq p$-subdivision of K_t, where $p = \max\{2, \frac{10}{\epsilon}\log\frac{1}{\epsilon}\} - 1$, hence

$$\mathrm{ex}(n, K_t^{(\leq p)}) \leq 2^{7t^2} n^{1+\frac{10\log p}{p}} \qquad \text{for } n > N(t, \epsilon).$$

Density and clique number of shallow subdivisions

Theorem (Kostochka, Pyber 1988)

Let $0 < \epsilon < 1$ *and* $t \in \mathbb{N}$. *Let* $p = \lfloor 1 + (4/\epsilon)(1 + 2\log t) \rfloor$. *Any graph* G *with* $\|G\| \geq 2^{2t(t-1)} t \cdot |G|^{1+\epsilon}$ *contains a* $\leq p$*-subdivision of* K_t.

Theorem (Jiang, 2009)

Let $0 < \epsilon < 1$ *and* $t \in \mathbb{N}$. *Then there exists* $N = N(\epsilon, t)$ *such that any graph* G *of order at least* N *with* $\|G\| \geq 2^{7t^2} \cdot |G|^{1+\epsilon}$ *contains a* $\leq p$*-subdivision of* K_t, *where* $p = \max\{2, \frac{10}{\epsilon} \log \frac{1}{\epsilon}\} - 1$, *hence*

$$\mathrm{ex}(n, K_t^{(\leq p)}) \leq 2^{7t^2} n^{1 + \frac{10 \log p}{p}} \qquad \text{for } n > N(t, \epsilon).$$

Density and chromatic number of shallow subdivisions

Lemma

Every graph G contains a subgraph with minimum degree at least $\chi(G)-1$.

Lemma (Dvořák, 2007)

Let $c \geq 4$ be an integer and let G be a graph with minimum degree $d > 56(c-1)^2 \frac{\log(c-1)}{\log c - \log(c-1)}$. Then the graph G contains a subgraph G' that is the 1-subdivision of a graph with chromatic number c.

Density and chromatic number of shallow subdivisions

Lemma

Every graph G contains a subgraph with minimum degree at least $\chi(G) - 1$.

Lemma (Dvořák, 2007)

Let $c \geq 4$ be an integer and let G be a graph with minimum degree $d > 56(c-1)^2 \frac{\log(c-1)}{\log c - \log(c-1)}$. Then the graph G contains a subgraph G' that is the 1-subdivision of a graph with chromatic number c.

Girth

1 Introduction

2 Density related parameters

3 Minors

4 Ordering and orientation related parameters

5 Homomorphism related parameters

6 Ramsey theory and extremal graph theory

7 Shallow subdivisions

8 Girth

Girth and Chromatic number: Erdős graphs

Theorem (Erdős, 1959)

For all integers c, g there exists a graph with girth at least g and chromatic number at least c.

Proof.

A random graph on n vertices and edge-probability $n^{(1-g)/g}$ has, with high probability, at most $n/2$ cycles of length at most g, but no independent set of size $n/2c$. Removing one vertex in each short cycle leaves a graph with girth at least g and chromatic number at least c. $\square$

Remark

This graph has $\Omega(n^{1+1/g})$ edges.

Girth and Chromatic number: Erdős graphs

Theorem (Erdős, 1959)

For all integers c, g there exists a graph with girth at least g and chromatic number at least c.

Proof.

A random graph on n vertices and edge-probability $n^{(1-g)/g}$ has, with high probability, at most $n/2$ cycles of length at most g, but no independent set of size $n/2c$. Removing one vertex in each short cycle leaves a graph with girth at least g and chromatic number at least c. $\square$

Remark

This graph has $\Omega(n^{1+1/g})$ edges.

Bollobás' construction

Theorem (Bollobás, 1978)

Let $k \geq 4$, $c = \{4k(1 + \log k)\}$ and $\Delta_0 = \Delta(k) = \{4ec\}$. Then for every sufficiently large n there is a graph G of order n such that $\Delta(G) \leq \Delta_0, \chi(G) \geq k$ and $g(G) \geq g_0$, where g_0 is s.t. $\frac{1}{g_0}(2c)^{g_0} < \frac{n}{12k^2}$.

Proof.

Let $\mathcal{C} = \{G : |G| = n, \|G\| = cn\}$ and $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \subseteq \mathcal{C}$ s.t.

In $\mathcal{C}_1$, $\exists F \subseteq E(G), |F| \leq n/k^2, \Delta(G - F) \leq \Delta_0$.

In $\mathcal{C}_2$, G has $\leq n/3k^2$ cycles of length $< g_0$.

In $\mathcal{C}_3$, $|W| = [n/(k-1)] \Rightarrow \|G[W]\| > [2n/3k^2]$.

Then, $|\mathcal{C}_1|, |\mathcal{C}_2|, |\mathcal{C}_3| \geq \frac{3}{4}|\mathcal{C}|$ hence $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3 \neq \emptyset$. □

Bollobás' construction

Theorem (Bollobás, 1978)

Let $k \geq 4$, $c = \{4k(1+\log k)\}$ and $\Delta_0 = \Delta(k) = \{4ec\}$. Then for every sufficiently large n there is a graph G of order n such that $\Delta(G) \leq \Delta_0, \chi(G) \geq k$ and $g(G) \geq g_0$, where g_0 is s.t. $\frac{1}{g_0}(2c)^{g_0} < \frac{n}{12k^2}$.

Proof.

Let $\mathcal{C} = \{G : |G| = n, \|G\| = cn\}$ and $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \subseteq \mathcal{C}$ s.t.

In $\mathcal{C}_1$, $\exists F \subseteq E(G), |F| \leq n/k^2, \Delta(G - F) \leq \Delta_0$.

In $\mathcal{C}_2$, G has $\leq n/3k^2$ cycles of length $< g_0$.

In $\mathcal{C}_3$, $|W| = [n/(k-1)] \Rightarrow \|G[W]\| > [2n/3k^2]$.

Then, $|\mathcal{C}_1|, |\mathcal{C}_2|, |\mathcal{C}_3| \geq \frac{3}{4}|\mathcal{C}|$ hence $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3 \neq \emptyset$. $\square$

What if a minor is excluded?

Theorem (Galluccio, Goddyn, Hell 2001)

Let H be a fixed graph.
Any H-minor free graph G of high enough girth admits a homomorphism to a large odd circuit.

Remark

Qualitative jump somewhere between proper minor closed classes and bounded degree classes.
Existence of high girth graphs with chromatic number $> 2 + \epsilon$ linked to existence of expanders?

What if a minor is excluded?

Theorem (Galluccio, Goddyn, Hell 2001)

Let H be a fixed graph.
Any H-minor free graph G of high enough girth admits a homomorphism to a large odd circuit.

Remark

Qualitative jump somewhere between proper minor closed classes and bounded degree classes.
Existence of high girth graphs with chromatic number $> 2 + \epsilon$ linked to existence of expanders?

A general case?

A conjecture of Erdős and Hajnal

For all integers c, g there exists an integer $f(c, g)$ such that every graph G of chromatic number at least $f(c, g)$ contains a subgraph of chromatic number at least c and girth at least g.

The case $g = 4$ was proved by Rödl, while the general case is still open.

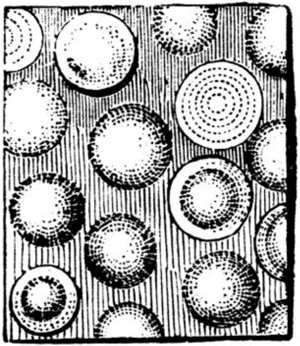
Girth and average degree

A conjecture of Thomassen

For all integers c, g there exists an integer $f(c, g)$ such that every graph G of average degree at least $f(c, g)$ contains a subgraph of average degree at least c and girth at least g.

- case $g = 4$: every graph can be made bipartite by deleting at most half of its edges (also kills odd g)
- case $g = 6$: Kuhn and Osthus (2002)

Relational Structures and First-Order Logic



Generalization

- Relational structures generalize graphs and directed graphs,...
- First order logic generalize subgraphs, homomorphisms from a template,...
  $\longrightarrow$ local properties
- Monadic second order logic generalize minors, colorings, homomorphisms to a template,...
  $\longrightarrow$ global properties

Generalization

- Relational structures generalize graphs and directed graphs,...
- First order logic generalize subgraphs, homomorphisms from a template,...
  $\longrightarrow$ local properties
- Monadic second order logic generalize minors, colorings, homomorphisms to a template,...
  $\longrightarrow$ global properties

Generalization

- Relational structures generalize graphs and directed graphs,...
- First order logic generalize subgraphs, homomorphisms from a template,...
  $\longrightarrow$ local properties
- Monadic second order logic generalize minors, colorings, homomorphisms to a template,...
  $\longrightarrow$ global properties

Relational Structures

A *relational vocabulary* σ is a finite set of relation symbols, each with a specified arity. A σ*-structure* $\mathbf{A}$ consists of a *universe* A, or *domain*, and an *interpretation* which associates to each relation symbol $R \in \sigma$ of some arity r, a relation $R^A \subseteq A^r$.

A σ-structure $\mathbf{B}$ is a *substructure* of $\mathbf{A}$ if $B \subseteq A$ and $R^B \subseteq R^A$ for every $R \in \sigma$. It is an *induced substructure* if $R^B = R^A \cap B^r$ for every $R \in \sigma$ of arity r.

A *homomorphism* $\mathbf{A} \to \mathbf{B}$ between two σ-structure is defined as a mapping $f : A \to B$ which satisfies for every relational symbol $R \in \sigma$ the following:

$$(x_1, \ldots, x_k) \in R^A \quad \Longrightarrow \quad (f(x_1), \ldots, f(x_k)) \in R^B.$$

The class (category) of all σ-structures is denoted by $\mathrm{Rel}(\sigma)$.

Relational Structures

A *relational vocabulary* σ is a finite set of relation symbols, each with a specified arity. A *σ-structure* $\mathbf{A}$ consists of a *universe* A, or *domain*, and an *interpretation* which associates to each relation symbol $R \in \sigma$ of some arity r, a relation $R^A \subseteq A^r$.
A σ-structure $\mathbf{B}$ is a *substructure* of $\mathbf{A}$ if $B \subseteq A$ and $R^B \subseteq R^A$ for every $R \in \sigma$. It is an *induced substructure* if $R^B = R^A \cap B^r$ for every $R \in \sigma$ of arity r.
A *homomorphism* $\mathbf{A} \to \mathbf{B}$ between two σ-structure is defined as a mapping $f : A \to B$ which satisfies for every relational symbol $R \in \sigma$ the following:

$$(x_1, \ldots, x_k) \in R^A \implies (f(x_1), \ldots, f(x_k)) \in R^B.$$

The class (category) of all σ-structures is denoted by $\mathrm{Rel}(\sigma)$.

Relational Structures

A *relational vocabulary* σ is a finite set of relation symbols, each with a specified arity. A *σ-structure* $\mathbf{A}$ consists of a *universe* A, or *domain*, and an *interpretation* which associates to each relation symbol $R \in \sigma$ of some arity r, a relation $R^A \subseteq A^r$.
A σ-structure $\mathbf{B}$ is a *substructure* of $\mathbf{A}$ if $B \subseteq A$ and $R^B \subseteq R^A$ for every $R \in \sigma$. It is an *induced substructure* if $R^B = R^A \cap B^r$ for every $R \in \sigma$ of arity r.
A *homomorphism* $\mathbf{A} \to \mathbf{B}$ between two σ-structure is defined as a mapping $f : A \to B$ which satisfies for every relational symbol $R \in \sigma$ the following:

$$(x_1, \ldots, x_k) \in R^A \quad \Longrightarrow \quad (f(x_1), \ldots, f(x_k)) \in R^B.$$

The class (category) of all σ-structures is denoted by $\mathrm{Rel}(\sigma)$.

First-Order Logic

- *atomic formulas, Boolean formulas, existential first-order formulas, first-order formulas.*
- The *quantifier count* $\mathrm{qcount}(\phi)$ of ϕ is the total number of quantifiers in ϕ.
- The *quantifier rank* $\mathrm{qrank}(\phi)$ of ϕ is the maximum nesting of quantifiers of its sub-formulas.

For a formula $\phi(x_1, \ldots, x_n)$ with free variables $x_1, \ldots, x_n$,

$$\mathbf{A} \models \phi(a_1, \ldots, a_n) \quad \Longleftrightarrow \quad \phi \text{ is true in } \mathbf{A} \text{ when } x_i \leftarrow a_i.$$

First-Order Logic

- *atomic formulas*, *Boolean formulas*, *existential first-order formulas*, *first-order formulas*.

- The *quantifier count* $\mathrm{qcount}(\phi)$ of ϕ is the total number of quantifiers in ϕ.
- The *quantifier rank* $\mathrm{qrank}(\phi)$ of ϕ is the maximum nesting of quantifiers of its sub-formulas.

For a formula $\phi(x_1, \ldots, x_n)$ with free variables $x_1, \ldots, x_n$,

$$\mathbf{A} \models \phi(a_1, \ldots, a_n) \quad \Longleftrightarrow \quad \phi \text{ is true in } \mathbf{A} \text{ when } x_i \leftarrow a_i.$$

First-Order Logic

- *atomic formulas*, *Boolean formulas*, *existential first-order formulas*, *first-order formulas*.

- The *quantifier count* $\mathrm{qcount}(\phi)$ of ϕ is the total number of quantifiers in ϕ.
- The *quantifier rank* $\mathrm{qrank}(\phi)$ of ϕ is the maximum nesting of quantifiers of its sub-formulas.

For a formula $\phi(x_1, \ldots, x_n)$ with free variables $x_1, \ldots, x_n$,

$$\mathbf{A} \models \phi(a_1, \ldots, a_n) \quad \Longleftrightarrow \quad \phi \text{ is true in } \mathbf{A} \text{ when } x_i \leftarrow a_i.$$

Ehrenfeucht-Fraïssé games

Ehrenfeucht-Fraïssé game $\Game_n(G, H)$: players Spoiler and Duplicator, played as follows:

- Start with $A_0 = B_0 = \emptyset$ and let π_0 be the empty mapping from A to B.
- For each $1 \leq i \leq n$, Spoiler picks either a vertex a in G or a vertex b in H.
  - In the first case, the Duplicator chooses a vertex b in H;
  - in the second case he chooses a vertex a in G.
- Let $A_i = A_{i-1} \cup \{a\}$ and $B_i = B_{i-1} \cup \{b\}$;
  - If no isomorphism $\pi_i : G[A_i] \to G[B_i]$ extending π_{i-1} exists such that $\pi(a) = b$ then Spoiler wins the game.
  - Otherwise, the game continues until $i = n$. If i reaches n and π_n is an isomorphism from $G[A_n]$ to $H[B_n]$ then Duplicator wins the game.

Ehrenfeucht-Fraïssé games

Ehrenfeucht-Fraïssé game $\Game_n(G,H)$: players Spoiler and Duplicator, played as follows:

- Start with $A_0 = B_0 = \emptyset$ and let π_0 be the empty mapping from A to B.
- For each $1 \leq i \leq n$, Spoiler picks either a vertex a in G or a vertex b in H.
  - In the first case, the Duplicator chooses a vertex b in H;
  - in the second case he chooses a vertex a in G.
- Let $A_i = A_{i-1} \cup \{a\}$ and $B_i = B_{i-1} \cup \{b\}$;
  - If no isomorphism $\pi_i : G[A_i] \to G[B_i]$ extending π_{i-1} exists such that $\pi(a) = b$ then Spoiler wins the game.
  - Otherwise, the game continues until $i = n$. If i reaches n and π_n is an isomorphism from $G[A_n]$ to $H[B_n]$ then Duplicator wins the game.

Ehrenfeucht-Fraïssé games

Ehrenfeucht-Fraïssé game $\Game_n(G, H)$: players Spoiler and Duplicator, played as follows:

- Start with $A_0 = B_0 = \emptyset$ and let π_0 be the empty mapping from A to B.
- For each $1 \leq i \leq n$, Spoiler picks either a vertex a in G or a vertex b in H.
  - In the first case, the Duplicator chooses a vertex b in H;
  - in the second case he chooses a vertex a in G.
- Let $A_i = A_{i-1} \cup \{a\}$ and $B_i = B_{i-1} \cup \{b\}$;
  - If no isomorphism $\pi_i : G[A_i] \to G[B_i]$ extending π_{i-1} exists such that $\pi(a) = b$ then Spoiler wins the game.
  - Otherwise, the game continues until $i = n$. If i reaches n and π_n is an isomorphism from $G[A_n]$ to $H[B_n]$ then Duplicator wins the game.

Ehrenfeucht-Fraïssé games

Ehrenfeucht-Fraïssé game $\Game_n(G, H)$: players Spoiler and Duplicator, played as follows:

- Start with $A_0 = B_0 = \emptyset$ and let π_0 be the empty mapping from A to B.
- For each $1 \leq i \leq n$, Spoiler picks either a vertex a in G or a vertex b in H.
  - In the first case, the Duplicator chooses a vertex b in H;
  - in the second case he chooses a vertex a in G.
- Let $A_i = A_{i-1} \cup \{a\}$ and $B_i = B_{i-1} \cup \{b\}$;
  - If no isomorphism $\pi_i : G[A_i] \to G[B_i]$ extending π_{i-1} exists such that $\pi(a) = b$ then Spoiler wins the game.
  - Otherwise, the game continues until $i = n$. If i reaches n and π_n is an isomorphism from $G[A_n]$ to $H[B_n]$ then Duplicator wins the game.

Ehrenfeucht-Fraïssé games

Ehrenfeucht-Fraïssé game $\eth_n(G,H)$: players Spoiler and Duplicator, played as follows:

- Start with $A_0 = B_0 = \emptyset$ and let π_0 be the empty mapping from A to B.
- For each $1 \leq i \leq n$, Spoiler picks either a vertex a in G or a vertex b in H.
  - In the first case, the Duplicator chooses a vertex b in H;
  - in the second case he chooses a vertex a in G.
- Let $A_i = A_{i-1} \cup \{a\}$ and $B_i = B_{i-1} \cup \{b\}$;
  - If no isomorphism $\pi_i : G[A_i] \to G[B_i]$ extending π_{i-1} exists such that $\pi(a) = b$ then Spoiler wins the game.
  - Otherwise, the game continues until $i = n$. If i reaches n and π_n is an isomorphism from $G[A_n]$ to $H[B_n]$ then Duplicator wins the game.

Ehrenfeucht-Fraïssé games

Ehrenfeucht-Fraïssé game $\eth_n(G,H)$: players Spoiler and Duplicator, played as follows:

- Start with $A_0 = B_0 = \emptyset$ and let π_0 be the empty mapping from A to B.
- For each $1 \leq i \leq n$, Spoiler picks either a vertex a in G or a vertex b in H.
  - In the first case, the Duplicator chooses a vertex b in H;
  - in the second case he chooses a vertex a in G.
- Let $A_i = A_{i-1} \cup \{a\}$ and $B_i = B_{i-1} \cup \{b\}$;
  - If no isomorphism $\pi_i : G[A_i] \to G[B_i]$ extending π_{i-1} exists such that $\pi(a) = b$ then Spoiler wins the game.
  - Otherwise, the game continues until $i = n$. If i reaches n and π_n is an isomorphism from $G[A_n]$ to $H[B_n]$ then Duplicator wins the game.

Ehrenfeucht-Fraïssé games

Ehrenfeucht-Fraïssé game $\eth_n(G, H)$: players Spoiler and Duplicator, played as follows:

- Start with $A_0 = B_0 = \emptyset$ and let π_0 be the empty mapping from A to B.
- For each $1 \leq i \leq n$, Spoiler picks either a vertex a in G or a vertex b in H.
  - In the first case, the Duplicator chooses a vertex b in H;
  - in the second case he chooses a vertex a in G.
- Let $A_i = A_{i-1} \cup \{a\}$ and $B_i = B_{i-1} \cup \{b\}$;
  - If no isomorphism $\pi_i : G[A_i] \to G[B_i]$ extending π_{i-1} exists such that $\pi(a) = b$ then Spoiler wins the game.
  - Otherwise, the game continues until $i = n$. If i reaches n and π_n is an isomorphism from $G[A_n]$ to $H[B_n]$ then Duplicator wins the game.

Back and Forth Equivalence

If Duplicator has a winning strategy for n then G and H are n-back and forth equivalent and we note $G \equiv^n H$.

Theorem (Fraïssé, Ehrenfeucht)

Two graphs (and more generally two structures) are n-back and forth equivalent if and only if they satisfy the same first order sentences of quantifier rank n.

Back and Forth Equivalence

If Duplicator has a winning strategy for n then G and H are n-back and forth equivalent and we note $G \equiv^n H$.

Theorem (Fraïssé, Ehrenfeucht)

Two graphs (and more generally two structures) are n-back and forth equivalent if and only if they satisfy the same first order sentences of quantifier rank n.

Interpretation (Model Theory)

Let $\mathcal{L}$ and $\mathcal{L}'$ be two languages and let T be a theory in $\mathcal{L}$. An *interpretation* I of $\mathcal{L}'$ in $\mathcal{L}$ is defined by:

- an integer n,
- an $\mathcal{L}$-formula $U[v_1, \ldots, v_n]$ with n free variables,
- an $\mathcal{L}$-formula $E[\overline{w}_1, \overline{w}_2]$ with $2n$ free variables ($\overline{w}_1, \overline{w}_2$ represent each a sequence of n variables),
- and an $\mathcal{L}$-formula $F_R[\overline{w}_1, \ldots, \overline{w}_k]$ with kn free variables for each relational symbol R with arity k,

which satisfy the following conditions:

1. the theory T entails that E is an equivalence relation;
2. the theory T entails that U is a union of E-classes;
3. for every integer k and every symbol R of arity k in $\mathcal{L}'$, T entails that F_R is interpreted by a set which is closed for the relation E.

Interpretation (Model Theory)

Let $\mathcal{L}$ and $\mathcal{L}'$ be two languages and let T be a theory in $\mathcal{L}$. An *interpretation* I of $\mathcal{L}'$ in $\mathcal{L}$ is defined by:

- an integer n,
- an $\mathcal{L}$-formula $U[v_1, \ldots, v_n]$ with n free variables,
- an $\mathcal{L}$-formula $E[\overline{w}_1, \overline{w}_2]$ with $2n$ free variables ($\overline{w}_1, \overline{w}_2$ represent each a sequence of n variables),
- and an $\mathcal{L}$-formula $F_R[\overline{w}_1, \ldots, \overline{w}_k]$ with kn free variables for each relational symbol R with arity k,

which satisfy the following conditions:

1. the theory T entails that E is an equivalence relation;
2. the theory T entails that U is a union of E-classes;
3. for every integer k and every symbol R of arity k in $\mathcal{L}'$, T entails that F_R is interpreted by a set which is closed for the relation E.

Interpretation (Model Theory)

If $\mathbf{A}$ is a model of T, we can *interpret* in $\mathbf{A}$ the $\mathcal{L}'$-structure $\mathbf{A}'$ defined as follows:

- the universe A' of $\mathbf{A}'$ is $U[\mathbf{A}]/E[\mathbf{A}]$;
- let R be a symbol of arity k of $\mathcal{L}'$ and $(a_1, \ldots, a_k) \in A'^k$; then $(a_1, \ldots, a_k) \in R^{A'}$ if and only if there exists $\overline{b}_1 \in a_1, \ldots, \overline{b}_k \in a_k$ such that $\mathbf{A} \models F_R[\overline{b}_1, \ldots, \overline{b}_k]$.

In such a case, $\mathbf{A}'$ is an *interpretation* of $\mathbf{A}$ by I, what we denote by $\mathbf{A}' = I(\mathbf{A})$.

Interpretation (Model Theory)

Lemma

For every formula $F[v_1, \ldots, v_k]$ *of* $\mathcal{L}'$ *there exists a formula* $I(f)[\overline{w}_1, \ldots, \overline{w}_k]$ *of* $\mathcal{L}$ *with kn free variables (each* $\overline{w}_i$ *represents a succession of n free variables) such that for every model* $\mathbf{A}$ *of* T*, if* $\mathbf{A}' = I(\mathbf{A})$ *and if* $(a_1, \ldots, a_k) \in A'^k$ *then the three following conditions are equivalent:*

1. $\mathbf{A}' \models F[a_1, \ldots, a_k]$;
2. *there exist* $\overline{b}_1 \in a_1, \ldots, \overline{b}_k \in a_k$ *such that* $\mathbf{A} \models I(F)[\overline{b}_1, \ldots, \overline{b}_k]$;
3. *for all* $\overline{b}_1 \in a_1, \ldots, \overline{b}_k \in a_k$ *it holds* $\mathbf{A} \models I(F)[\overline{b}_1, \ldots, \overline{b}_k]$.

Preservation theorems

Theorem (Łoś-Tarski theorem)

a first-order formula is preserved under extensions on all structures if, and only if, it is logically equivalent to an existential formula.

Theorem (Lyndon's theorem)

a first-order formula is preserved under surjective homomorphisms on all structures if, and only if, it is logically equivalent to a positive formula.

Theorem (Homomorphism Preservation Theorem)

a first-order formula is preserved under homomorphisms on all structures if, and only if, it is logically equivalent to an existential positive formula.

Preservation theorems

Theorem (Łoś-Tarski theorem)

a first-order formula is preserved under extensions on all structures if, and only if, it is logically equivalent to an existential formula.

Theorem (Lyndon's theorem)

a first-order formula is preserved under surjective homomorphisms on all structures if, and only if, it is logically equivalent to a positive formula.

Theorem (Homomorphism Preservation Theorem)

a first-order formula is preserved under homomorphisms on all structures if, and only if, it is logically equivalent to an existential positive formula.

Preservation theorems

Theorem (Łoś-Tarski theorem)

a first-order formula is preserved under extensions on all structures if, and only if, it is logically equivalent to an existential formula.

Theorem (Lyndon's theorem)

a first-order formula is preserved under surjective homomorphisms on all structures if, and only if, it is logically equivalent to a positive formula.

Theorem (Homomorphism Preservation Theorem)

a first-order formula is preserved under homomorphisms on all structures if, and only if, it is logically equivalent to an existential positive formula.

Preservation theorems

Łoś-Tarski theorem

False in the finite.

Theorem (Lyndon's theorem)

a first-order formula is preserved under surjective homomorphisms on all structures if, and only if, it is logically equivalent to a positive formula.

Theorem (Homomorphism Preservation Theorem)

a first-order formula is preserved under homomorphisms on all structures if, and only if, it is logically equivalent to an existential positive formula.

Preservation theorems

Łoś-Tarski theorem

False in the finite.

Lyndon's theorem

False in the finite.

Theorem (Homomorphism Preservation Theorem)

a first-order formula is preserved under homomorphisms on all structures if, and only if, it is logically equivalent to an existential positive formula.

Preservation theorems

Łoś-Tarski theorem

False in the finite.

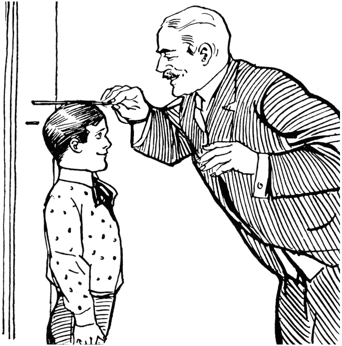
Lyndon's theorem

False in the finite.

Rossman, 2006

a first-order formula is preserved under homomorphisms on finite structures if, and only if, it is logically equivalent on the finite to an existential positive formula.

Measuring sparsity



Shallow minors

9 Shallow minors

10 Grad and top-grad

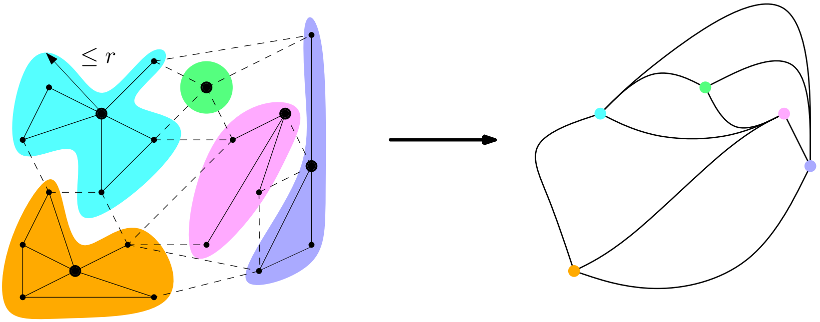
11 Stability by lexicographic product

12 Immersions

Shallow minors

Depth r minor

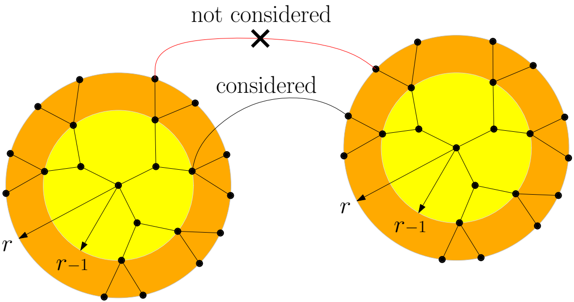
$V_1, \ldots, V_p$ disjoint subsets of $V(G)$ such that each $G[V_i]$ is connected and has radius at most r.
$V(H) = \{v_1, \ldots, v_p\}$, $E(H) = \{\{v_i, v_j\} : \omega(V_i, V_j) \neq \emptyset\}$.



Shallow minors

Depth $r - 1/2$ minor

$V_1, \ldots, V_p$ disjoint subsets of $V(G)$ such that $v_i \in V_i$, each $G[V_i]$ is connected and $\mathrm{dist}(v_i, v) \le r$ $(\forall v \in V_i)$.
$V(H) = \{v_1, \ldots, v_p\}$ and $E(H) = \{\{v_i, v_j\}$ s.t. $\exists \{x_i, x_j\} \in E(G)$ with $x_i \in V_i, x_j \in V_j, \mathrm{dist}(v_i, x_i) + \mathrm{dist}(x_j, v_j) < 2r - 1$.



Shallow minors

$$G \in G \triangledown 0 \subseteq G \triangledown \frac{1}{2} \subseteq G \triangledown 1 \subseteq \cdots \subseteq G \triangledown a \subseteq \ldots G \triangledown \infty$$

$\triangledown$ -arithmetic

Let a, b be half-integers and let c be the half-integer defined by

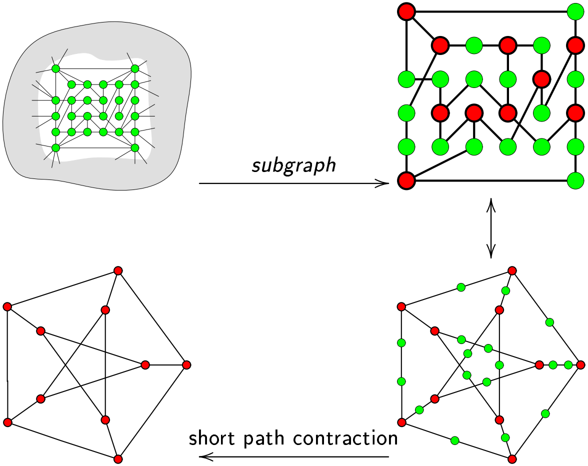
$$(2c+1) = (2a+1)(2b+1).$$

Then for every graph G:

$$G \triangledown ((\lceil a \rceil + 1)b) \subseteq (G \triangledown a) \triangledown b \subseteq G \triangledown c$$

$$G \triangledown q \subseteq ((\ldots(G \underbrace{\triangledown 1) \triangledown 1) \ldots) \triangledown 1}_{q \text{ times}} \subseteq G \triangledown \left(\frac{3^q - 1}{2}\right).$$

Shallow topological minors



Shallow topological minors

$$
\begin{array}{ccccccccccccc}
 & & G \nabla 0 & \subseteq & G \nabla 1/2 & \subseteq & \dots & G \nabla a & \subseteq & \dots & \subseteq & G \nabla \infty \\
 & & \| & & \cup\!\vert & & & \cup\!\vert & & & & \cup\!\vert \\
G & \in & G \,\widetilde{\nabla}\, 0 & \subseteq & G \,\widetilde{\nabla}\, 1/2 & \subseteq & \dots & G \,\widetilde{\nabla}\, a & \subseteq & \dots & \subseteq & G \,\widetilde{\nabla}\, \infty
\end{array}
$$

$\widetilde{\nabla}$-arithmetic

Let a, b, c be such that $(2c+1) = (2a+1)(2b+1)$.

$$(G \,\widetilde{\nabla}\, a) \,\widetilde{\nabla}\, b = G \,\widetilde{\nabla}\, c$$

$$((\dots(G \underbrace{\widetilde{\nabla}\, 1) \,\widetilde{\nabla}\, 1) \dots) \,\widetilde{\nabla}\, 1}_{q \text{ times}} = G \,\widetilde{\nabla} \left(\frac{3^q - 1}{2}\right).$$

Grad and top-grad

9 Shallow minors

10 Grad and top-grad

11 Stability by lexicographic product

12 Immersions

grad and top-grad

The *greatest reduced average density (grad)* with rank r of a graph G is defined by

$$\nabla_r(G) = \max\left\{\frac{\|H\|}{|H|} : H \in G \mathbin{\triangledown} r\right\}$$

The *top-grad* with rank r of G is defined by

$$\widetilde{\nabla}_r(G) = \max\left\{\frac{\|H\|}{|H|} : H \in G \mathbin{\widetilde{\triangledown}} r\right\}$$

grad and top-grad

$$\begin{array}{ccccccccccc}
 & & \nabla_0(G) & \leq & \nabla_{1/2}(G) & \leq & \dots & \leq & \nabla_\infty(G) & = & \nabla(G) \\
 & & \| & & \rotatebox{90}{$\leq$} & & & & \rotatebox{90}{$\leq$} & & \\
\frac{\mathrm{mad}(G)}{2} & = & \widetilde{\nabla}_0(G) & \leq & \widetilde{\nabla}_{1/2}(G) & \leq & \dots & \leq & \widetilde{\nabla}_\infty(G) & = & \widetilde{\nabla}(G)
\end{array}$$

$$\frac{\mathrm{h}(G)-1}{2} \leq \nabla(G) = O(\mathrm{h}(G)\sqrt{\log \mathrm{h}(G)}) \qquad (\mathrm{h} \asymp \nabla)$$

$$\frac{\mathrm{h}_t(G)-1}{2} \leq \widetilde{\nabla}(G) = O(\mathrm{h}_t(G)^2) \qquad (\mathrm{h}_t \asymp \widetilde{\nabla})$$

$\widetilde{\nabla}_r \asymp \nabla_r$

Theorem (Dvořák, 2007)

Let $r, d \geq 1$ be integers and let $p = 4(4d)^{(r+1)^2}$. If $\nabla_r(G) \geq p$, then G contains a subgraph F' that is a $\leq 2r$-subdivision of a graph F with minimum degree d.

Corollary

For every graph G and every integer $r \geq 1$ holds

$$\widetilde{\nabla}_r(G) \leq \nabla_r(G) \leq 4(4\widetilde{\nabla}_r(G))^{(r+1)^2}$$

Stability by lexicographic product

9 Shallow minors

10 Grad and top-grad

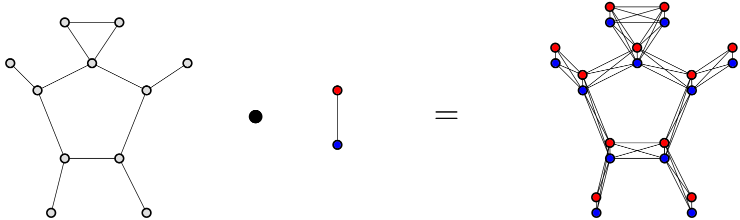
11 Stability by lexicographic product

12 Immersions

The lexicographic product

Let G and H be graphs. The *lexicographic product* $G \bullet H$ is defined by

$$V(G \bullet H) = V(G) \times V(H)$$
$$E(G \bullet H) = \{\{(x,y),(x',y')\} : \{x,x'\} \in E(G) \text{ or } x = x' \text{ and } \{y,y'\} \in E(H)\}.$$



Stability of top-grads

Theorem

Let G be a graph, let $p \geq 2$ be a positive integer and let r be a half-integer. Then

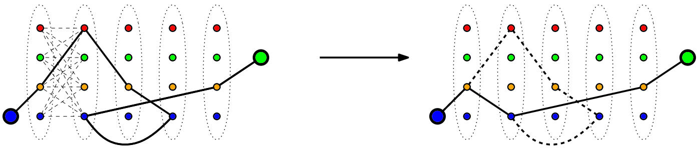
$$\widetilde{\nabla}_r(G \bullet K_p) \leq \max(2r(p-1)+1, p^2)\widetilde{\nabla}_r(G) + p - 1$$

Proof

Let $V(K_p) = \{a_1, a_2, \ldots, a_p\}$.
Vertices (v, a_i) and (v, a_j) are *twins* in $G \bullet K_p$, v is their projection on G.
Let $H \in (G \bullet K_p) \,\widetilde{\triangledown}\, r$ be such that $\frac{\|H\|}{|H|} = \widetilde{\nabla}_r(G \bullet K_p)$ and let $S(H) \subseteq G \bullet K_p$ be the corresponding $\leq 2r$-subdivision of H in G.
Then...

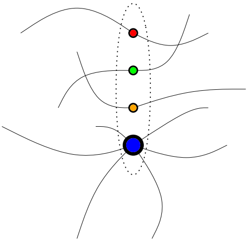
Proof (Cont'd)

1. We may assume that no branch of $S(H)$ contains two twin vertices, except if the branch is a single edge path linking two twin vertices:



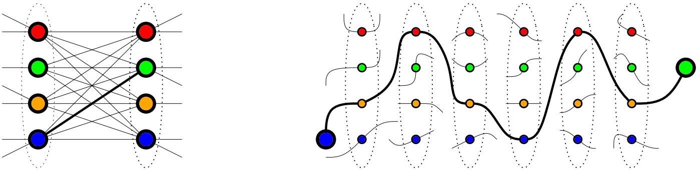
Proof (Cont'd)

2. Start with $H_1 = H$ and $S(H_1) = S(H)$.
$\forall$ subdivision vertex $v \in S(H_1)$ which twin is a principal vertex of $S(H)$, delete the branch of $S(H_1)$ and the corresponding edge of H_1.
At most $(p-1)|H|$ edges deleted $\Longrightarrow$ $\frac{\|H_1\|}{|H_1|} \geq \frac{\|H\|}{|H|} - (p-1)$.



Proof (Cont'd)

3. Conflict graph C of H_1: $V(C) = E(H_1)$, $E(C) =$ set of $\{e_1, e_2\}$ such that:
   - either e_1 and e_2 are not subdivided in $S(H_1)$ and their endpoints are equal or twins,
   - or e_1 and e_2 are subdivided in $S(H_1)$ and one of the subdivision vertices of the branch corresponding to e_1 is a twin of one of the subdivision vertex of the branch corresponding to e_2.



$$\implies \quad \chi(C) \leq \Delta(C) + 1 \leq \max(p^2, 2(p-1)r + 1)$$

Proof (End)

4. Let H_2 be a partial graph of H_1 defined by a monochromatic set of edges of H_1 of size at least $\frac{\|H_1\|}{\max(2r(p-1)+1, p^2)}$. Let v be a principal vertex of $S(H_2)$. Then two edges incident to v cannot have their other endpoints equal or twins (because of the coloration).
5. Let H_3 be the projection of H_2 on G. Because of the coloration, no two edges of H_2 are projected on a same edge of H_3 and only the edges linking twin vertices may have been removed (simultaneously to the removal of one all but one of the twins). As the surplus twins then have degree at most $p-1$ they can be removed safely. Then $\widetilde{\nabla}_r(G) \geq \frac{\|H_3\|}{|H_3|} \geq \frac{\|H_2\|}{|H_2|}$ and the result follows. $\square$

Proof (End)

4. Let H_2 be a partial graph of H_1 defined by a monochromatic set of edges of H_1 of size at least $\frac{\|H_1\|}{\max(2r(p-1)+1,p^2)}$. Let v be a principal vertex of $S(H_2)$. Then two edges incident to v cannot have their other endpoints equal or twins (because of the coloration).
5. Let H_3 be the projection of H_2 on G. Because of the coloration, no two edges of H_2 are projected on a same edge of H_3 and only the edges linking twin vertices may have been removed (simultaneously to the removal of one all but one of the twins). As the surplus twins then have degree at most $p-1$ they can be removed safely. Then $\widetilde{\nabla}_r(G) \geq \frac{\|H_3\|}{|H_3|} \geq \frac{\|H_2\|}{|H_2|}$ and the result follows. $\square$

Immersions

9 Shallow minors

10 Grad and top-grad

11 Stability by lexicographic product

12 Immersions

Immersions

An *immersion* of a graph H in a graph G is a function ι with domain $V(H) \cup E(H)$, such that:

- $\iota(v) \in V(G)$ for all $v \in V(H)$, and $\iota(u) \neq \iota(v)$ for all distinct $u, v \in V(H)$;
- for each edge $e = \{u, v\}$ of H, $\iota(e)$ is a path of G with ends $\iota(u), \iota(v)$;
- for all distinct $e, f \in E(H)$, $E(\iota(e)) \cap E(\iota(f)) = \emptyset$.

Alternatively, a graph H is an immersion of the graph G if H can be obtained from G by a sequence of vertex deletions, edge deletions and *edge lift* (An edge lift consists in replacing a pair of adjacent edges $\{u, v\}$ and $\{v, w\}$ by a single edge $\{u, w\}$).

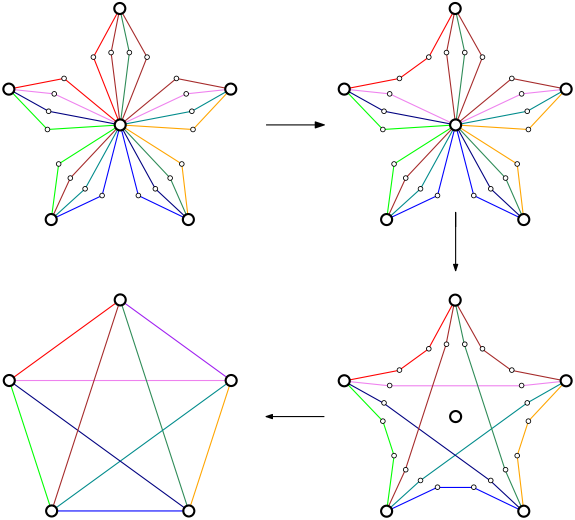
Immersions

An *immersion* of a graph H in a graph G is a function ι with domain $V(H) \cup E(H)$, such that:

- $\iota(v) \in V(G)$ for all $v \in V(H)$, and $\iota(u) \neq \iota(v)$ for all distinct $u, v \in V(H)$;
- for each edge $e = \{u, v\}$ of H, $\iota(e)$ is a path of G with ends $\iota(u), \iota(v)$;
- for all distinct $e, f \in E(H)$, $E(\iota(e)) \cap E(\iota(f)) = \emptyset$.

Alternatively, a graph H is an immersion of the graph G if H can be obtained from G by a sequence of vertex deletions, edge deletions and *edge lift* (An edge lift consists in replacing a pair of adjacent edges $\{u, v\}$ and $\{v, w\}$ by a single edge $\{u, w\}$).

Edge lifts



Shallow immersions

- *stretch*: maximum of $(\|\iota(e)\| - 1)/2$;
- *complexity*: maximum of $|\{e : v \in \iota(e)\}| + |\{x : v = \iota(x)\}|$.
- *shallow immersion* of *depth* (p, q): immersion of stretch at most q and complexity at most p.

→ immersions at depth $(1, p)$ = topological minors at depth p,

→ every graph may be immersed into a very sparse graph with a stretch of 3/2 if one does not bound the complexity of the immersion.

Shallow immersions

- *stretch*: maximum of $(\|\iota(e)\| - 1)/2$;
- *complexity*: maximum of $|\{e : v \in \iota(e)\}| + |\{x : v = \iota(x)\}|$.
- *shallow immersion* of *depth* (p, q): immersion of stretch at most q and complexity at most p.

→ immersions at depth $(1, p)$ = topological minors at depth p,

→ every graph may be immersed into a very sparse graph with a stretch of $3/2$ if one does not bound the complexity of the immersion.

Shallow immersions

Let $G \overset{\propto}{\nabla} (p,q)$ be the class of all shallow immersions of G with complexity p and stretch q. Then

$$G \mathbin{\widetilde{\nabla}} q \subseteq G \overset{\propto}{\nabla} (p,q) \subseteq (G \bullet \overline{K}_p) \mathbin{\widetilde{\nabla}} q.$$

The *imm-grad* of rank (p,q) of G is

$$\overset{\propto}{\nabla}_{p,q}(G) = \max_{H \in G \overset{\propto}{\nabla} (p,q)} \frac{\|H\|}{|H|}.$$

If P is a polynomial then all of ∇_r, $\widetilde{\nabla}_r$ and $\overset{\propto}{\nabla}_{P(r),r}$ are polynomially equivalent:

$$\nabla_r \asymp \widetilde{\nabla}_r \asymp \overset{\propto}{\nabla}_{P(r),r}.$$

Shallow immersions

Let $G \overset{\propto}{\triangledown} (p,q)$ be the class of all shallow immersions of G with complexity p and stretch q. Then

$$G \mathbin{\widetilde{\triangledown}} q \subseteq G \overset{\propto}{\triangledown} (p,q) \subseteq (G \bullet \overline{K_p}) \mathbin{\widetilde{\triangledown}} q.$$

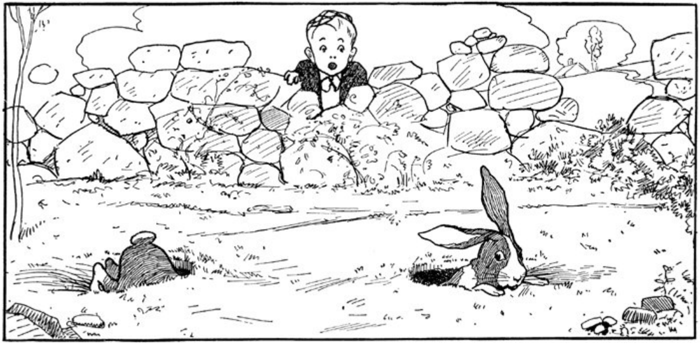
The *imm-grad* of rank (p,q) of G is

$$\overset{\propto}{\nabla}_{p,q}(G) = \max_{H \in G \overset{\propto}{\triangledown} (p,q)} \frac{\|H\|}{|H|}.$$

If P is a polynomial then all of $\nabla_r, \widetilde{\nabla}_r$ and $\overset{\propto}{\nabla}_{P(r),r}$ are polynomially equivalent:

$$\nabla_r \asymp \widetilde{\nabla}_r \asymp \overset{\propto}{\nabla}_{P(r),r}.$$

Taxonomy of classes



Class resolutions

13 Class resolutions

14 Supremum limits and bounds

15 Class taxonomy

Class resolutions

$$\mathcal{C} \triangledown a = \bigcup_{G\in\mathcal{C}} G \triangledown a, \quad \mathcal{C} \,\widetilde{\triangledown}\, a = \bigcup_{G\in\mathcal{C}} G \,\widetilde{\triangledown}\, a, \quad \mathcal{C} \,\overset{\propto}{\triangledown}\, (a,b) = \bigcup_{G\in\mathcal{C}} G \,\overset{\propto}{\triangledown}\, (a,b)$$

- $\mathcal{C} \triangledown 0$: *monotone closure* of $\mathcal{C}$,
- $\mathcal{C} \triangledown \infty$:*minor closure*, $\mathcal{C} \,\widetilde{\triangledown}\, \infty$: *topological closure*.

- *resolution*: $\mathcal{C}^{\triangledown} = (\mathcal{C} \triangledown 0, \dots, \mathcal{C} \triangledown a, \dots)$
- *topological resolution*: $\mathcal{C}^{\widetilde{\triangledown}} = (\mathcal{C} \,\widetilde{\triangledown}\, 0, \dots, \mathcal{C} \,\widetilde{\triangledown}\, a, \dots)$
- *immersion resolution*: $\mathcal{C}^{\overset{\propto}{\triangledown}} = (\mathcal{C} \,\overset{\propto}{\triangledown}\, (0,1), \dots, \mathcal{C} \,\widetilde{\triangledown}\, (a, a+1), \dots)$

Class resolution in time

$$\mathcal{C} \subseteq \mathcal{C} \triangledown 0 \subseteq \mathcal{C} \triangledown 1 \subseteq \dots \subseteq \mathcal{C} \triangledown t \subseteq \dots \subseteq \mathcal{C} \triangledown \infty$$

$$\xrightarrow[\textit{time}]{\hspace{6cm}}$$

Class resolutions

$$\mathcal{C} \triangledown a = \bigcup_{G \in \mathcal{C}} G \triangledown a, \quad \mathcal{C} \,\widetilde{\triangledown}\, a = \bigcup_{G \in \mathcal{C}} G \,\widetilde{\triangledown}\, a, \quad \mathcal{C} \overset{\propto}{\triangledown} (a,b) = \bigcup_{G \in \mathcal{C}} G \overset{\propto}{\triangledown} (a,b)$$

- $\mathcal{C} \triangledown 0$: *monotone closure* of $\mathcal{C}$,
- $\mathcal{C} \triangledown \infty$:*minor closure*, $\mathcal{C} \,\widetilde{\triangledown}\, \infty$: *topological closure*.

- *resolution*: $\mathcal{C}^{\triangledown} = (\mathcal{C} \triangledown 0, \dots, \mathcal{C} \triangledown a, \dots)$
- *topological resolution*: $\mathcal{C}^{\widetilde{\triangledown}} = (\mathcal{C} \,\widetilde{\triangledown}\, 0, \dots, \mathcal{C} \,\widetilde{\triangledown}\, a, \dots)$
- *immersion resolution*: $\mathcal{C}^{\overset{\propto}{\triangledown}} = (\mathcal{C} \overset{\propto}{\triangledown} (0,1), \dots, \mathcal{C} \,\widetilde{\triangledown}\, (a, a+1), \dots)$

Class resolution in time

$$\mathcal{C} \subseteq \mathcal{C} \triangledown 0 \subseteq \mathcal{C} \triangledown 1 \subseteq \dots \subseteq \mathcal{C} \triangledown t \subseteq \dots \subseteq \mathcal{C} \triangledown \infty$$

$$\xrightarrow[\textit{time}]{\hspace{8em}}$$

Supremum limits and bounds

13 Class resolutions

14 Supremum limits and bounds

15 Class taxonomy

Supremum limits

Let $\mathcal{C}$ be an infinite class of graphs, let $f : \mathcal{C} \to \mathbb{R}$ be a graph invariant, and let $\mathrm{Inj}(\mathbb{N}, \mathcal{C})$ = injective mappings from $\mathbb{N}$ to $\mathcal{C}$.

$$\limsup_{G \in \mathcal{C}} f(G) = \sup_{\phi \in \mathrm{Inj}(\mathbb{N}, \mathcal{C})} \limsup_{i \to \infty} f(\phi(i))$$

$\rightarrow$ $\limsup_{G \in \mathcal{C}} f(G)$ exists and $\in \mathbb{R} \cup \{-\infty, +\infty\}$.

Resolution limits:

$$\lim_{i \to \infty} \limsup_{G \in \mathcal{C} \triangledown i} f(G), \qquad \lim_{i \to \infty} \limsup_{G \in \mathcal{C} \widetilde{\triangledown} i} f(G)$$

Supremum limits

Let $\mathcal{C}$ be an infinite class of graphs, let $f : \mathcal{C} \to \mathbb{R}$ be a graph invariant, and let $\mathrm{Inj}(\mathbb{N}, \mathcal{C})$ = injective mappings from $\mathbb{N}$ to $\mathcal{C}$.

$$\limsup_{G \in \mathcal{C}} f(G) = \sup_{\phi \in \mathrm{Inj}(\mathbb{N}, \mathcal{C})} \limsup_{i \to \infty} f(\phi(i))$$

$\to$ $\limsup_{G \in \mathcal{C}} f(G)$ exists and $\in \mathbb{R} \cup \{-\infty, +\infty\}$.

Resolution limits:

$$\lim_{i \to \infty} \limsup_{G \in \mathcal{C} \,\triangledown\, i} f(G), \qquad \lim_{i \to \infty} \limsup_{G \in \mathcal{C} \,\widetilde{\triangledown}\, i} f(G)$$

Class taxonomy

13 Class resolutions

14 Supremum limits and bounds

15 Class taxonomy

Classification by resolutions

Direct classification of classes by a resolution $\mathfrak{C} \in \{\mathcal{C}^{\nabla}, \mathcal{C}^{\widetilde{\nabla}}, \mathcal{C}^{\overset{\propto}{\nabla}}\}$:

- If there exist finite a such that $\mathfrak{C}_a = \mathcal{G}raph$, $\mathcal{C}$ is *somewhere dense*.
- If $\mathfrak{C}_a \neq \mathcal{G}raph$ for every a, $\mathcal{C}$ is *nowhere dense*.

Lemma

Let G be a graph and let a be a half-integer. Then

$$\omega(G \mathbin{\widetilde{\nabla}} a) \leq \omega(G \mathbin{\nabla} a) \leq 2\omega(G \mathbin{\widetilde{\nabla}} (3a+1))^{\lfloor a \rfloor + 1}$$

Corollary

Resolutions $\mathcal{C}^{\nabla}, \mathcal{C}^{\widetilde{\nabla}}$ and $\mathcal{C}^{\overset{\propto}{\nabla}}$ define the same classification.

Classification by resolutions

Direct classification of classes by a resolution $\mathfrak{C} \in \{\mathcal{C}^{\nabla}, \mathcal{C}^{\widetilde{\nabla}}, \mathcal{C}^{\overset{\propto}{\nabla}}\}$:

- If there exist finite a such that $\mathfrak{C}_a = \mathcal{G}raph$, $\mathcal{C}$ is *somewhere dense*.
- If $\mathfrak{C}_a \neq \mathcal{G}raph$ for every a, $\mathcal{C}$ is *nowhere dense*.

Lemma

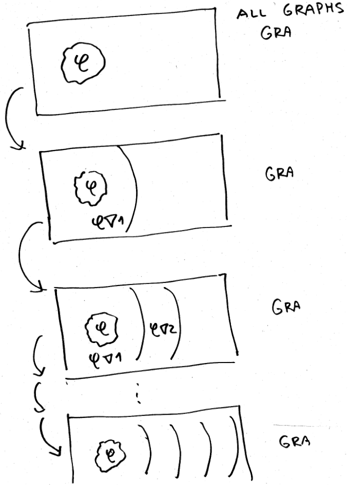
Let G be a graph and let a be a half-integer. Then

$$\omega(G \mathbin{\widetilde{\nabla}} a) \leq \omega(G \mathbin{\nabla} a) \leq 2\omega(G \mathbin{\widetilde{\nabla}} (3a+1))^{\lfloor a \rfloor + 1}$$

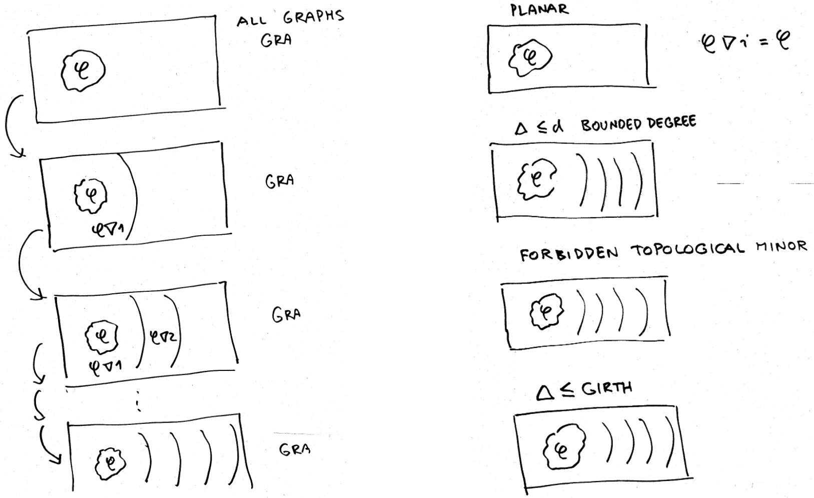
Corollary

Resolutions $\mathcal{C}^{\nabla}, \mathcal{C}^{\widetilde{\nabla}}$ and $\mathcal{C}^{\overset{\propto}{\nabla}}$ define the same classification.

Filtration by time (of appearance)



Filtration by time (of appearance)



Classification by logarithmic density

Theorem (Class trichotomy)

Let $\mathcal{C}$ be an infinite class of graphs and $\mathfrak{C} \in \{\mathcal{C}^{\triangledown}, \mathcal{C}^{\widetilde{\triangledown}}, \mathcal{C}^{\overset{\propto}{\triangledown}}\}$. Then the limit

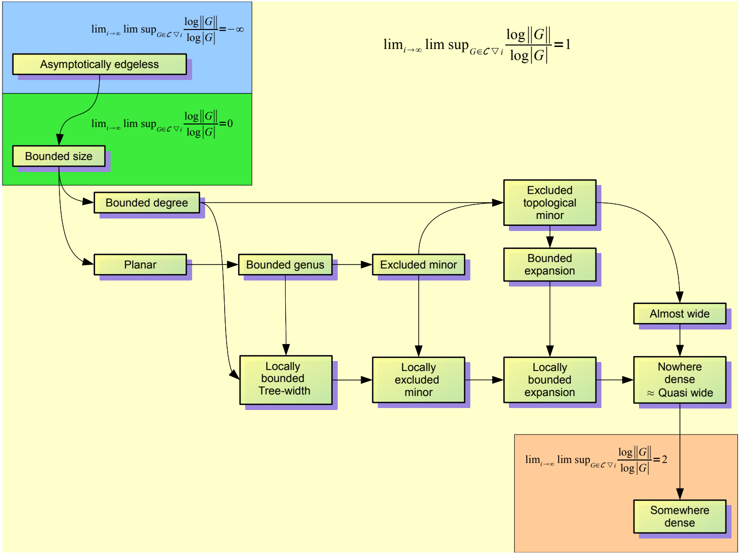
$$\lim_{r\to\infty} \limsup_{G\in\mathfrak{C}_r} \frac{\log \|G\|}{\log |G|}$$

can only take four values, namely $-\infty, 0, 1$ or 2.
The class $\mathcal{C}$ is:

- *a* bounded size *class if and only if* 0 *or* $-\infty$*,*
- *a* nowhere dense *class if and only if* $\in \{-\infty, 0, 1\}$*,*
- *a* somewhere dense *class if and only if* 2*.*

Notice that $\mathcal{C}^{\triangledown}, \mathcal{C}^{\widetilde{\triangledown}}$ and $\mathcal{C}^{\overset{\propto}{\triangledown}}$ define the same trichotomy.

The world

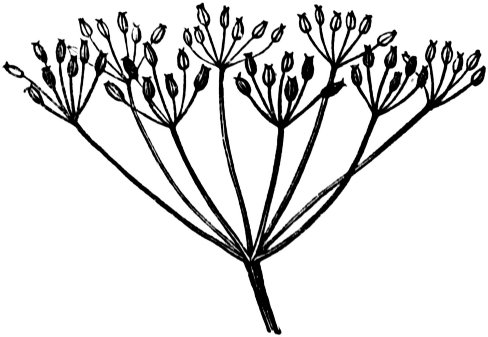

$\lim_{i\to\infty} \limsup_{G\in\mathcal{C}\nabla i} \frac{\log\|G\|}{\log|G|} = -\infty$
Asymptotically edgeless
$\lim_{i\to\infty} \limsup_{G\in\mathcal{C}\nabla i} \frac{\log\|G\|}{\log|G|} = 0$
Bounded size
$\lim_{i\to\infty} \limsup_{G\in\mathcal{C}\nabla i} \frac{\log\|G\|}{\log|G|} = 1$
Bounded degree
Excluded topological minor
Planar
Bounded genus
Excluded minor
Bounded expansion
Almost wide
Locally bounded Tree-width
Locally excluded minor
Locally bounded expansion
Nowhere dense ≈ Quasi wide
$\lim_{i\to\infty} \limsup_{G\in\mathcal{C}\nabla i} \frac{\log\|G\|}{\log|G|} = 2$
Somewhere dense

Characterization of nowhere dense classes

Let $\mathcal{C}$ be an unbounded size infinite class of graphs, let F be a graph with at least one edge and let q be a positive integer. Then the following conditions are equivalent:

1. $\mathcal{C}$ is a class of nowhere dense graphs,
2. for every integer r, $\mathcal{C} \mathbin{\nabla} r$ is not the class of all finite graphs,
3. for every integer r, $\mathcal{C} \mathbin{\widetilde{\nabla}} r$ is not the class of all finite graphs,
4. $\mathcal{C}$ is a uniformly quasi-wide class,
5. $H(\mathcal{C})$ is a quasi-wide class,
6. $\lim_{r\to\infty} \limsup_{G \in \mathcal{C} \nabla r} \frac{\log \|G\|}{\log |G|} = 1$,
7. $\lim_{r\to\infty} \limsup_{G \in \mathcal{C} \widetilde{\nabla} r} \frac{\log \|G\|}{\log |G|} = 1$,
8. $\lim_{r\to\infty} \limsup_{G \in \mathcal{C}} \frac{\log \nabla_r(G)}{\log |G|} = 0$,
9. $\lim_{r\to\infty} \limsup_{G \in \mathcal{C}} \frac{\log \widetilde{\nabla}_r(G)}{\log |G|} = 0$,
10. $\lim_{p\to\infty} \limsup_{G \in \mathcal{C}} \frac{\log \chi_p(G)}{\log |G|} = 0$,
11. $\lim_{i\to\infty} \limsup_{G \in \mathcal{C} \nabla i} \frac{\log \chi(G)}{\log |G|} = 0$,
12. $\lim_{p\to\infty} \limsup_{G \in \mathcal{C}} \frac{\log \mathrm{col}_p(G)}{\log |G|} = 0$,
13. $\lim_{p\to\infty} \limsup_{G \in \mathcal{C}} \frac{\log \mathrm{wcol}_p(G)}{\log |G|} = 0$,
14. for every integer c, the class $\mathcal{C} \bullet K_c = \{G \bullet K_c : G \in \mathcal{C}\}$ is a class of nowhere dense graphs,
15. $\lim_{i\to\infty} \limsup_{G \in \mathcal{C} \nabla i} \frac{\log(\#F \subseteq G)}{\log |G|} < |F|$,
16. for every polynomial P, the class $\mathcal{C}'$ of the 1-transitive fraternal augmentations of directed graphs $\vec{G}$ with $\Delta^-(\vec{G}) \leq P(\nabla_0(G))$ and $G \in \mathcal{C}$ form a class of nowhere dense graphs.

Tree-Depth and Decomposition



Definition and basic properties of tree-depth

16 Definition and basic properties of tree-depth

17 Relations with other invariants

18 Finiteness and Consequences

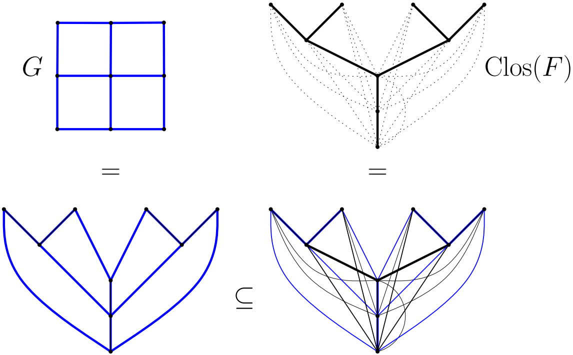
19 Low tree-depth decompositions

20 Fraternal augmentation

21 The weak coloring approach

Definition

The *tree-depth* of G is the minimum height of a rooted forest F such that $G \subseteq \mathrm{Clos}(F)$.



- See also *rank function*, *vertex ranking number*, the minimum height of an *elimination tree*

Basic properties

- If G is connected with DFS-tree Y then

$$\mathrm{height}(Y) \geq \mathrm{td}(G) \geq \log_2(\mathrm{height}(Y)+1).$$

- td is a minor monotone invariant.

Recursive definition

$$\mathrm{td}(G) = \begin{cases} 1, & \text{if } |G| = 1; \\ 1 + \min_{v \in V(G)} \mathrm{td}(G - v), & \text{if } G \text{ is connected and } |G| > 1; \\ \max_{i=1,\dots,p} \mathrm{td}(G_i), & \text{otherwise;} \end{cases}$$

(where $G_1, \dots, G_p$ are the connected components of G).

Basic properties

- If G is connected with DFS-tree Y then

$$\mathrm{height}(Y) \geq \mathrm{td}(G) \geq \log_2(\mathrm{height}(Y)+1).$$

- td is a minor monotone invariant.

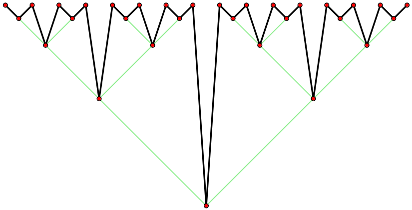
Recursive definition

$$\mathrm{td}(G) = \begin{cases} 1, & \text{if } |G| = 1; \\ 1 + \min_{v \in V(G)} \mathrm{td}(G - v), & \text{if } G \text{ is connected and } |G| > 1; \\ \max_{i=1,\dots,p} \mathrm{td}(G_i), & \text{otherwise}; \end{cases}$$

(where $G_1, \dots, G_p$ are the connected components of G).

Tree-depth and paths

The tree-depth of a path of order n is $\mathrm{td}(P_n) = \lceil \log_2(n+1) \rceil$.



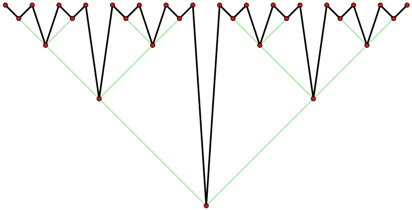
$\mathcal{C}$ has bounded tree-depth,

$\iff$ $\mathcal{C}$ excludes some path P_n as a subgraph (as a minor),

$\iff$ $\mathcal{C}$ is degenerate and excludes some path P_n as an induced subgraph.

Tree-depth and paths

The tree-depth of a path of order n is $\mathrm{td}(P_n) = \lceil \log_2(n+1) \rceil$.



$\mathcal{C}$ has bounded tree-depth,

$\iff$ $\mathcal{C}$ excludes some path P_n as a subgraph (as a minor),

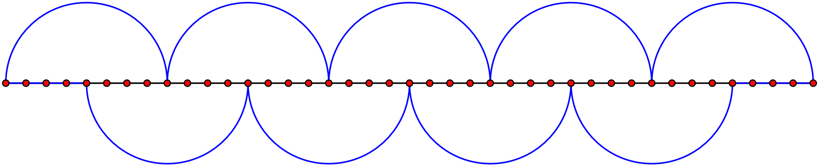
$\iff$ $\mathcal{C}$ is degenerate and excludes some path P_n as an induced subgraph.

Tree-depth and cycles

Lemma

Let G be a biconnected graph, and let L be the length of a longest cycle of G. Then

$$1 + \lceil \log_2 L \rceil \leq \mathrm{td}(G) \leq 1 + (L-2)^2.$$



Scattered sets

Lemma

Let $\mathcal{C}$ be a hereditary class of graphs. Then the two following properties are equivalent:

(i) $\exists s$ *and* $N : \mathbb{N} \to \mathbb{N}$ *such that* $\forall p$ *and* $\forall G \in \mathcal{C}$ *of order at least* $N(p)$, $\exists S \subseteq V(G)$ *with* $|S| \leq s$ *so that* $G - S$ *has at* $\geq p$ *connected components.*

(ii) $\mathcal{C}$ *has bounded tree-depth*

Relations with other invariants

16 Definition and basic properties of tree-depth

17 Relations with other invariants

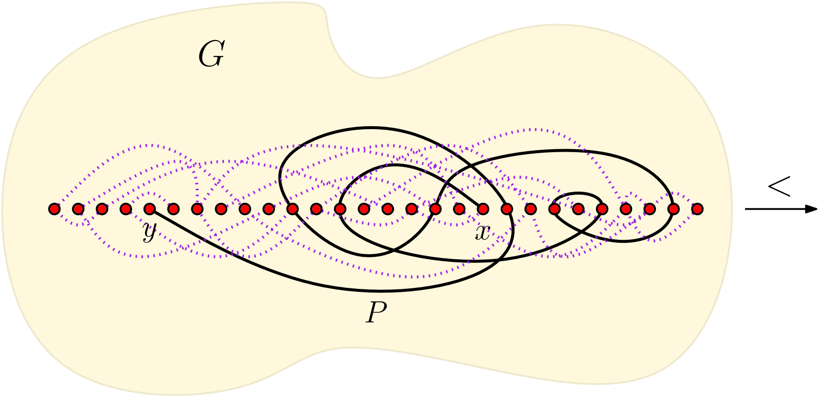
18 Finiteness and Consequences

19 Low tree-depth decompositions

20 Fraternal augmentation

21 The weak coloring approach

Weak coloring numbers



$$\mathrm{td}(G) = \mathrm{wcol}_\infty(G)$$

Tree-width

$$\mathrm{tw}(G) \le \mathrm{pw}(G) \le \mathrm{td}(G) \le \mathrm{tw}(G)\,\log|G|.$$

Lemma

Let $0 < \alpha < 1$ and let $\mathcal{C}$ be an hereditary class of graphs such that each graph $G \in \mathcal{C}$ of order n has tree-width at most Cn^α. Then, every graph $G \in \mathcal{C}$ of order n has tree-depth at most $\frac{C}{1-2^{-\alpha}}n^\alpha$.

Corollary

Every graph G of order n with no minor isomorphic to K_h has tree-depth at most $(2+\sqrt{2})\sqrt{h^3 n}$.

Tree-width

$$\mathrm{tw}(G) \le \mathrm{pw}(G) \le \mathrm{td}(G) \le \mathrm{tw}(G) \log |G|.$$

Lemma

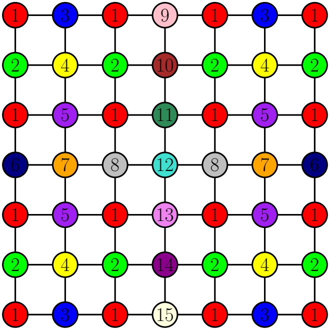
Let $0 < \alpha < 1$ and let $\mathcal{C}$ be an hereditary class of graphs such that each graph $G \in \mathcal{C}$ of order n has tree-width at most Cn^α. Then, every graph $G \in \mathcal{C}$ of order n has tree-depth at most $\frac{C}{1-2^{-\alpha}} n^\alpha$.

Corollary

Every graph G of order n with no minor isomorphic to K_h has tree-depth at most $(2+\sqrt{2})\sqrt{h^3 n}$.

Centered coloring

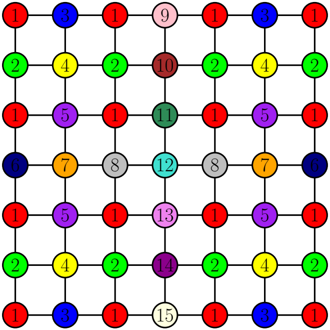
A *centered coloring* of a graph G is a vertex coloring such that, for any (induced) connected subgraph H, some color $c(H)$ appears exactly once in H.



→ For every graph G, $\mathrm{td}(G)$ is the minimum number of colors in a centered coloring of G.

Centered coloring

A *centered coloring* of a graph G is a vertex coloring such that, for any (induced) connected subgraph H, some color $c(H)$ appears exactly once in H.



$\rightarrow$ For every graph G, $\mathrm{td}(G)$ is the minimum number of colors in a centered coloring of G.

Finiteness and Consequences

16 Definition and basic properties of tree-depth

17 Relations with other invariants

18 Finiteness and Consequences

19 Low tree-depth decompositions

20 Fraternal augmentation

21 The weak coloring approach

Colored trees

Let $r_c(n)$ be the number of c-colored unlabeled rooted trees of order n. Then

$$r_c(n) \lesssim \alpha(c)(c\alpha(c)/A)^n n^{-3/2}$$

Define $F(c,t)$ by inductively by:

$$F(c,t) = \begin{cases} c, & \text{if } t = 1, \\ \sum_{i=1}^{F(c,t-1)+1} r_c(i), & \text{otherwise.} \end{cases}$$

Lemma

Let F be a c-colored rooted forest. If $\mathrm{height}(F) = t$ and $|F| > F(c,t)$ then F has an involutive automorphism exchanging two branches or two rooted trees.

Graphs with bounded tree-depth

Theorem

Any c-colored graph G of order $n > F(c, \mathrm{td}(G))$ has a non-trivial involuting color-preserving automorphism $\mu : G \to G$ which reverses no edge.

Corollary

Any asymmetric graph of tree-depth t has order at most $F(1, t)$.

Corollary

For any c-colored graph G, $\exists A \subseteq V(G)$, $|A| \leq F(c, t)$, such that $G \to G[A]$.

First-order properties

In general, a graph is not n-equivalent to one of its proper subgraph. However:

Theorem

For every integers t, n, c there exists an integer $N(t, n, c)$ such that every graph with tree-depth at most t with vertices colored using c colors is n-equivalent to one of its induced subgraphs of order at most $N(t, n, c)$:

$$\forall G\ \exists A \subseteq V(G),\ |A| \leq N(\mathrm{td}(G), n, c) \text{ and } G \equiv^n G[A].$$

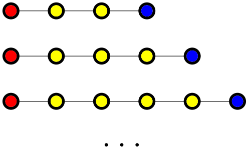
Well quasi orders

Let $(Q, \leq)$ be a well quasi-ordered set and let t be an integer. Denote by $\mathcal{T}_t(Q)$ the class of Q-labeled graphs of tree-depth at most t.
Define $G \subseteq_i H$ if $\exists f : V(G) \to V(H)$ such that $G \cong H[f(V(G))]$ and $\mathrm{label}(f(x)) \geq \mathrm{label}(x)$ for every $x \in V(G)$.

Lemma (Ding, 1992)

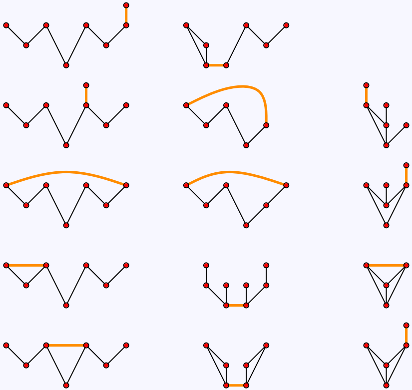
The class $\mathcal{T}_t(Q)$ is well quasi ordered by $\subseteq_i$.

Remark: the class of 3-colored paths does not have such a property.



Forbidden subgraph characterization

Obstructions for $\mathrm{td} \leq 3$ (Dvořák)



Let $\mathcal{T}_t^{(c)}$ = c-colored graphs of tree-depth $\leq t$.
Define $H \subseteq_i^\star G$ if $\exists A:\ H \cong G[A]$ and $G \to H$ (color preserving).

Theorem

The class $\mathcal{T}_t^{(c)}$ is well quasi ordered by $\subseteq_i^\star$.

Proof.

Consider a subset S of $\mathcal{T}_t^{(c)}$. Then S contains finitely many classes of graphs which are homomorphically equivalent. In each class, there are finitely many graphs which are minimal for $\subseteq_i$. $\square$

Low tree-depth decompositions

16 Definition and basic properties of tree-depth

17 Relations with other invariants

18 Finiteness and Consequences

19 Low tree-depth decompositions

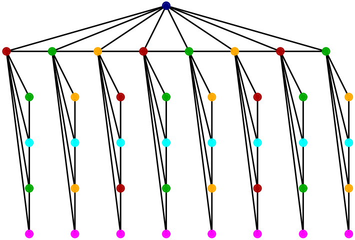
20 Fraternal augmentation

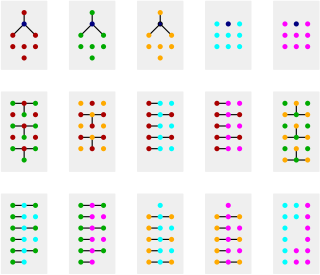
21 The weak coloring approach

Low tree-depth decompositions

Chromatic numbers χ_p

$\chi_p(G)$ is the minimum of colors such that any subset I of $\leq p$ colors induce a subgraph G_I so that $\mathrm{td}(G_I) \leq |I|$.

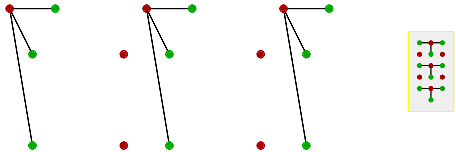




Low tree-depth decompositions

Chromatic numbers χ_p

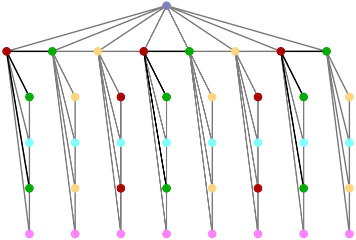
$\chi_p(G)$ is the minimum of colors such that any subset I of $\leq p$ colors induce a subgraph G_I so that $\mathrm{td}(G_I) \leq |I|$.

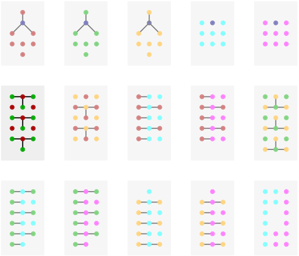


Low tree-depth decompositions

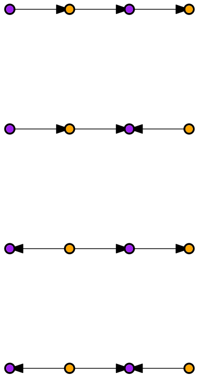
Chromatic numbers χ_p

$\chi_p(G)$ is the minimum of colors such that any subset I of $\leq p$ colors induce a subgraph G_I so that $\mathrm{td}(G_I) \leq |I|$.

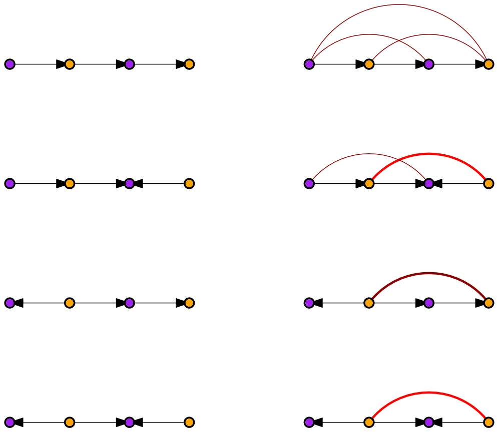




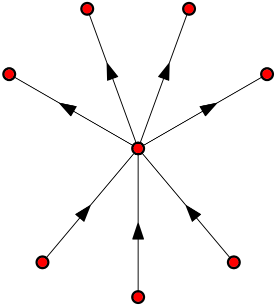
χ_2: coloration with no 2-colored P_4's

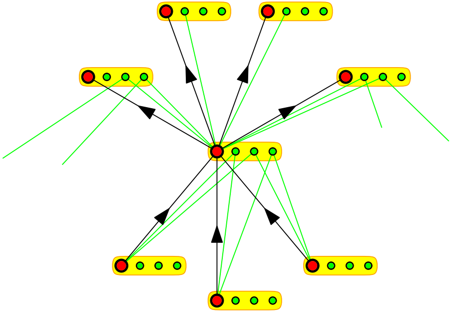


χ_2: coloration with no 2-colored P_4's



Fraternal Augmentation





Fraternal augmentation

16 Definition and basic properties of tree-depth

17 Relations with other invariants

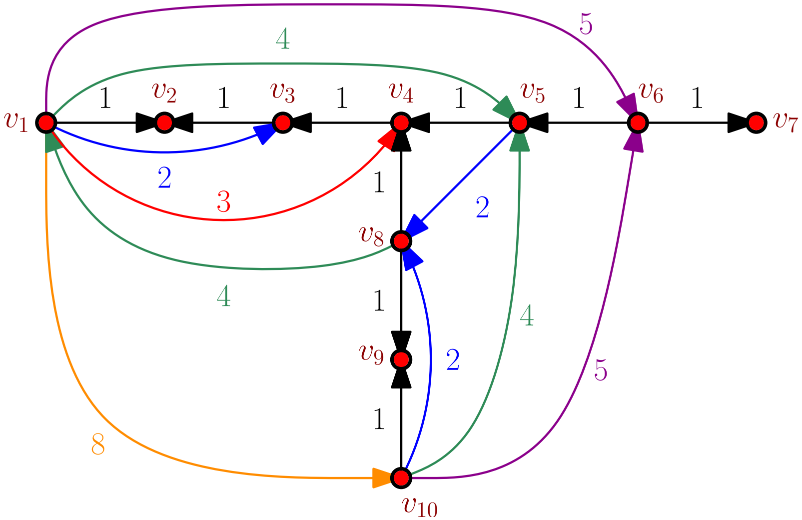
18 Finiteness and Consequences

19 Low tree-depth decompositions

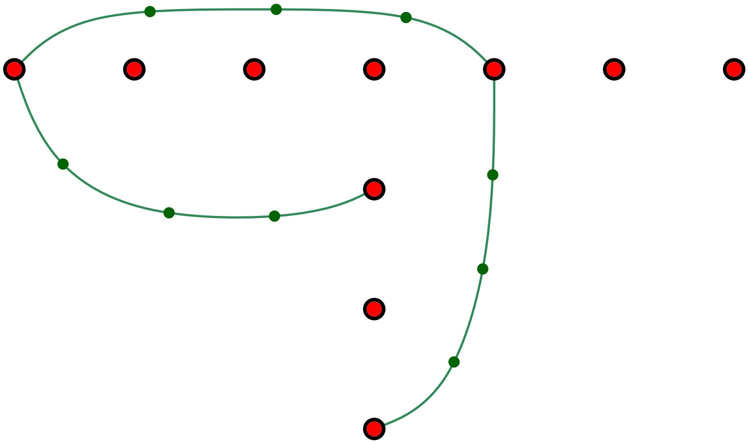
20 Fraternal augmentation

21 The weak coloring approach

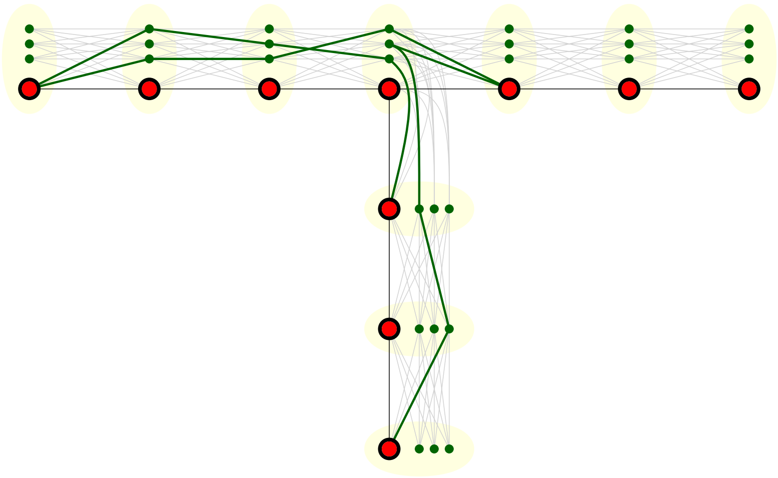
Fraternal augmentation



Embedding in $G \bullet \overline{K}_p$



Embedding in $G \bullet \overline{K}_p$



Fraternal augmentation and centered coloring

Let $k = 2^{p-1} + 2$. Let c be a coloring such that $c(x) \neq c(y)$ if there exists in $\vec{G}_{\leq k}$ a directed path from x to y of length at most p.

Lemma

Let P be a p-colored path in G and let V_P be the vertex set of P. Then the length of P is at most $2^p - 2$, and there exists a vertex $s \in P$ such that every other vertex $v \in P$ may be reached from s by a directed path of $\vec{G}_{\leq k}[V(P)]$.

Fraternal augmentation and centered coloring

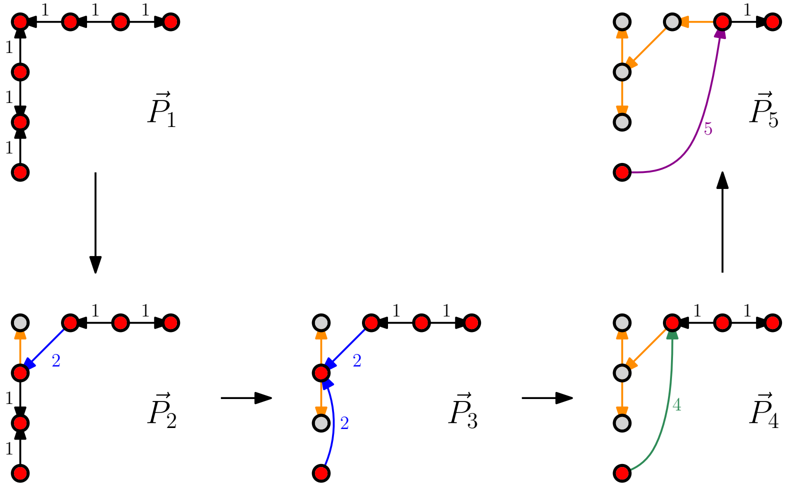
Proof.

Let L be the length of P. Define inductively $\vec{P}_i$ in $\vec{G}_{\leq k}[V(P)]$ of length $L_i = L + 1 - i$ such that

- every vertex of $V(P) \setminus \vec{P}_i$ can be reached from a vertex of $\vec{P}_i$ by a directed path of $\vec{G}_{\leq k}[V(P)]$;
- for every adjacent vertices u, v of $\vec{P}_i$ there exists in $\vec{G}_{\leq k}[V(P)]$ a directed path of length $\lceil \log_2(w(u,v)) \rceil$ starting from v which intersects $\vec{P}_i$ only at v.

$\square$

Fraternal augmentation and centered coloring



Low tree-depth decomposition

Theorem

Let $\vec{G}$ be a directed graph. Define $A_1 = 0, B_1 = \Delta^-(\vec{G})$ and inductively:

$$A_i = \sum_{j=2}^{i-1} A_j B_{i-j} + \frac{1}{2}\sum_{j=1}^{i-1} B_j B_{i-j}$$

$$B_i = \max((i-1)A_i + 1, (A_i+1)^2)\widetilde{\nabla}_{(i-1)/2}(G) + A_i$$

Then, $\forall p \geq 2,\ \chi_p(G) \leq 1 + 2\sum_{i=1}^{2^{p-1}+2} B_i$. Hence $\chi_p(G)$ is bounded by a polynomial $P_p(\widetilde{\nabla}_{2^{p-2}+1/2}(G))$, where P_p has degree about 2^{2^p}.

The weak coloring approach

16 Definition and basic properties of tree-depth

17 Relations with other invariants

18 Finiteness and Consequences

19 Low tree-depth decompositions

20 Fraternal augmentation

21 The weak coloring approach

The weak coloring approach

Theorem (Zhu, 2008)

Let G be a graph, let $k \in \mathbb{N}$ and let $p = (k-1)/2$.

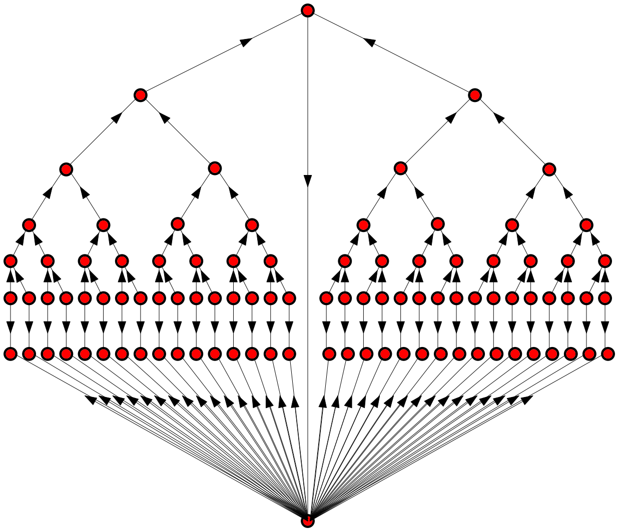
- $\nabla_p(G) + 1 \leq \mathrm{wcol}_k(G)$,
- *If* $\nabla_p(G) \leq m$ *then* $\mathrm{col}_k(G) \leq 1 + q_k$, *where* q_k *is defined as* $q_1 = 2m$ *and for* $i \geq 1$, $q_{i+1} = q_1 q_i^{2i^2}$.

As $\mathrm{col}_k(G) \leq \mathrm{wcol}_k(G) \leq \mathrm{col}_k(G)^k$ *(Kierstead, 2003) we have* $\nabla_p \asymp \mathrm{col}_k \asymp \mathrm{wcol}_k$.

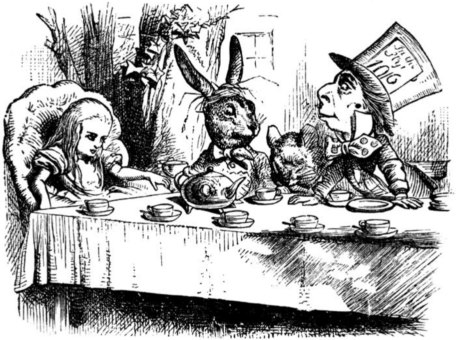
Theorem (Zhu, 2008)

If G is a graph with $\mathrm{wcol}_{2^{p-2}}(G) \leq m$, then G has a p-centered coloring using at most m colors.

Difference between the two approaches



Independence and First-Order Logic



Wideness of a class

22 Wideness of a class

23 Finding d-independent sets

24 Characterizations

25 Applications

26 Homomorphism preservation theorems

Scattered sets

Definition

- A subset X of $V(G)$ is *r-independent* if $\mathrm{dist}_G(x,y) > r$ for every $x, \neq y \in X$.
- The *r-independence number* of G is the maximum size $\alpha_r(G)$ of an r-independent set of G.

→ If $\Delta(\mathcal{C}) < \infty$ then $\forall d$ it holds:

$$\liminf_{G \in \mathcal{C}} \alpha_d(G) = \infty$$

→ If $\mathrm{td}(\mathcal{C}) < \infty$ then there exists $s \in \mathbb{N}$ such that:

$$\liminf_{G \in \mathcal{C}} \max_{|S| \leq s} \alpha_\infty(G - S) = \infty$$

Scattered sets

Definition

- A subset X of $V(G)$ is *r-independent* if $\mathrm{dist}_G(x, y) > r$ for every $x, \neq y \in X$.
- The *r-independence number* of G is the maximum size $\alpha_r(G)$ of an r-independent set of G.

$\rightarrow$ If $\Delta(\mathcal{C}) < \infty$ then $\forall d$ it holds:

$$\liminf_{G \in \mathcal{C}} \alpha_d(G) = \infty$$

$\rightarrow$ If $\mathrm{td}(\mathcal{C}) < \infty$ then there exists $s \in \mathbb{N}$ such that:

$$\liminf_{G \in \mathcal{C}} \max_{|S| \leq s} \alpha_\infty(G - S) = \infty$$

Scattered sets

Definition

- A subset X of $V(G)$ is *r-independent* if $\mathrm{dist}_G(x,y) > r$ for every $x, \neq y \in X$.
- The *r-independence number* of G is the maximum size $\alpha_r(G)$ of an r-independent set of G.

$\rightarrow$ If $\Delta(\mathcal{C}) < \infty$ then $\forall d$ it holds:

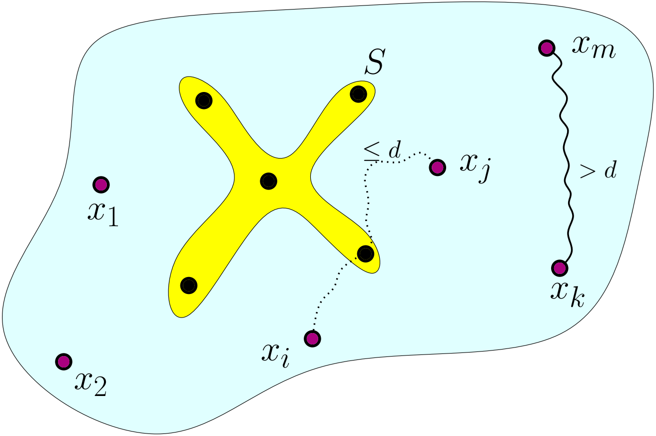
$$\liminf_{G \in \mathcal{C}} \alpha_d(G) = \infty$$

$\rightarrow$ If $\mathrm{td}(\mathcal{C}) < \infty$ then there exists $s \in \mathbb{N}$ such that:

$$\liminf_{G \in \mathcal{C}} \max_{|S| \leq s} \alpha_\infty(G - S) = \infty$$

The function $\Phi_{\mathcal{C}}$

$$\Phi_{\mathcal{C}}(d) = \min\Big\{s : \liminf_{G \in \mathcal{C}} \max_{|S| \le s} \alpha_d(G - S) = \infty\Big\}.$$



Wideness of a class

Definition (Dawar)

A class $\mathcal{C}$ is

- *wide* if $\forall d,\ \Phi_{\mathcal{C}}(d) = 0$,
- *almost wide* if $\sup_d \Phi_{\mathcal{C}}(d) < \infty$,
- *quasi wide* if $\forall d,\ \Phi_{\mathcal{C}}(d) < \infty$.

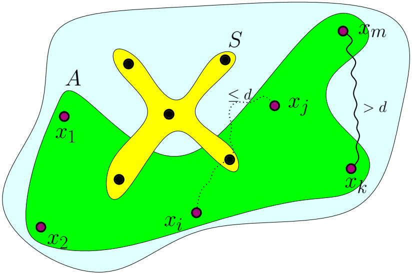
Remark

This is not a hereditary notion.

Uniform version: the function $\overline{\Phi}_{\mathcal{C}}$

Let $\alpha_r(G|A)$ be the maximum size of an r-independent set of G included in A.

$$\overline{\Phi}_{\mathcal{C}}(d) = \min\Big\{s : \liminf_{N\to\infty} \inf_{\substack{G\in\mathcal{C} \\ A\in\binom{V(G)}{N}}} \max_{|S|\leq s} \alpha_d(G-S|A) = \infty\Big\}.$$



Wideness of a class

Definition (Dawar)

A class $\mathcal{C}$ is

- *uniformly wide* if $\forall d,\ \overline{\Phi}_{\mathcal{C}}(d) = 0$,
- *uniformly almost wide* if $\sup_d \overline{\Phi}_{\mathcal{C}}(d) < \infty$,
- *uniformly quasi wide* if $\forall d,\ \overline{\Phi}_{\mathcal{C}}(d) < \infty$.

Remark

Invariant by monotone closure: $\overline{\Phi}_{\mathcal{C}} = \overline{\Phi}_{\mathcal{C}\,\triangledown\,0}$.

Wide classes

Theorem

Let $\mathcal{C}$ be a hereditary class of graphs. Then the following are equivalent:

- $\Phi_{\mathcal{C}}(2) = 0$,
- $\overline{\Phi}_{\mathcal{C}}(2) = 0$,
- $\Delta(\mathcal{C}) < \infty$,
- *$\mathcal{C}$ is wide,*
- *$\mathcal{C}$ is uniformly wide.*

Finding d-independent sets

22 Wideness of a class

23 Finding d-independent sets

24 Characterizations

25 Applications

26 Homomorphism preservation theorems

Finding 1-independent sets

This is Ramsey theorem!

Let c, n be integers.
Every graph of order at least $R(c, n)$ contains

- either a clique of size c,
- or an independent set of size n.

Corollary

Let $\mathcal{C}$ be a hereditary class.

- Either $\omega(\mathcal{C}) = \infty$ and $\Phi_{\mathcal{C}}(1) = \overline{\Phi}_{\mathcal{C}}(1) = \infty$,
- or $\omega(\mathcal{C}) < \infty$ and $\Phi_{\mathcal{C}}(1) = \overline{\Phi}_{\mathcal{C}}(1) = 0$.

Finding 2-independent sets

Let $R^\star(p,q,n) = R(\overbrace{q,q,\ldots,q}^{\binom{n-1}{2} \text{ times}},p)$.

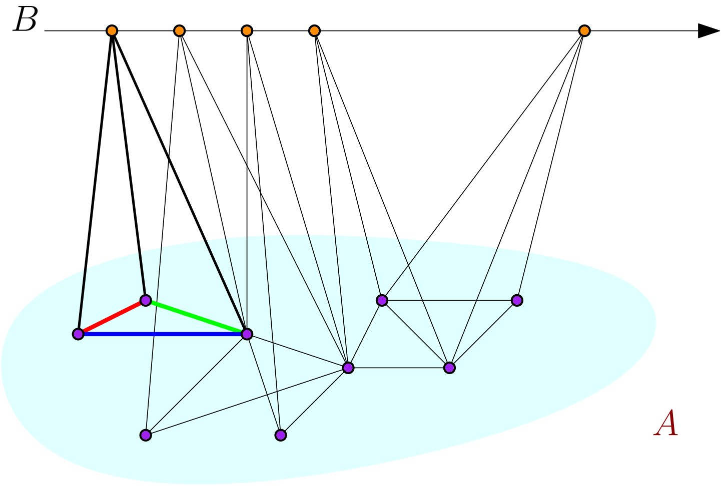
Lemma

Let $G = (A \cup B, E)$ be a bipartite graph.

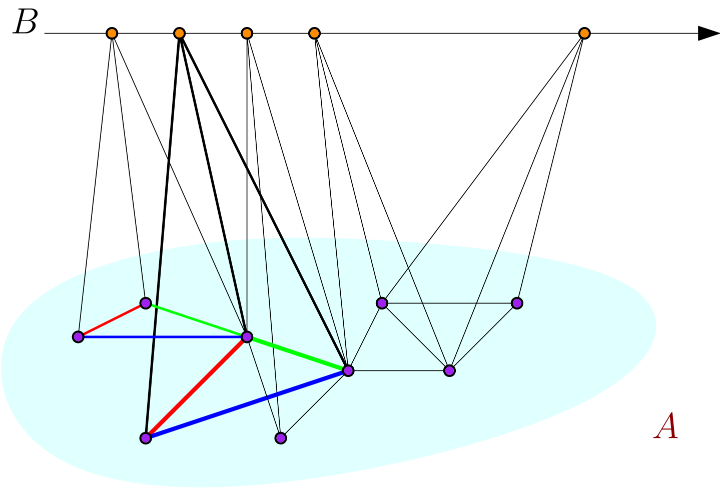
If $|A| \geq R^\star(p,q,n)$ then at least one of the following holds:

- *A includes a 2-independent set of size p;*
- *A includes the principal vertices of a $K_q^{(1)}$;*
- *B includes a vertex of degree $\geq n$.*

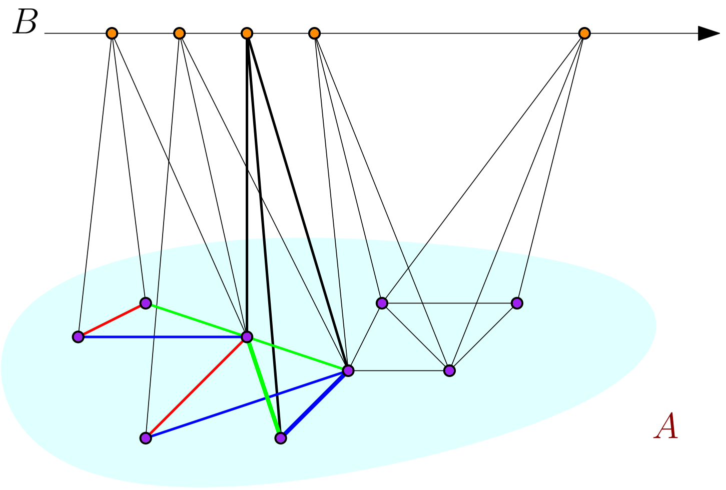
Proof



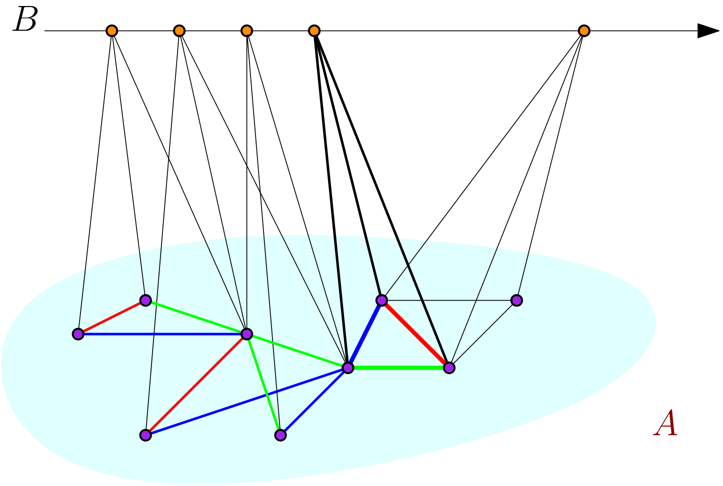
Proof



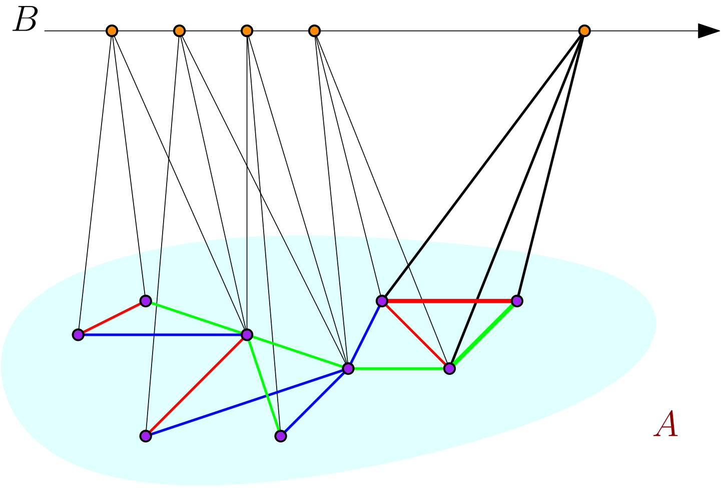
Proof



Proof



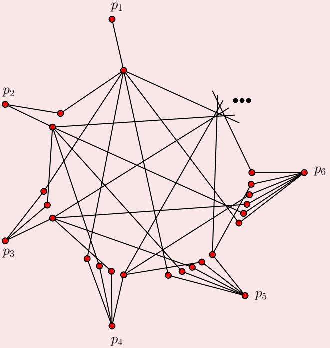
Proof



A first consequence

$$\omega\left(G \,\widetilde{\nabla}\, \frac{1}{2}\right) \geq f\left(\omega\left(G \,\nabla\, \frac{1}{2}\right)\right).$$

Does not generalize: $\omega(G_i \,\widetilde{\nabla}\, 1) = 2$ but $\omega(G_i \,\nabla\, 1) \to \infty$



Finding 2-independent sets

$$\text{Let } \Theta(m,a,b,s) = \begin{cases} R^\star(m,a,b), & \text{if } s = 0; \\ R^\star(m,a,\Theta(m,a,b,s-1)), & \text{otherwise.} \end{cases}.$$

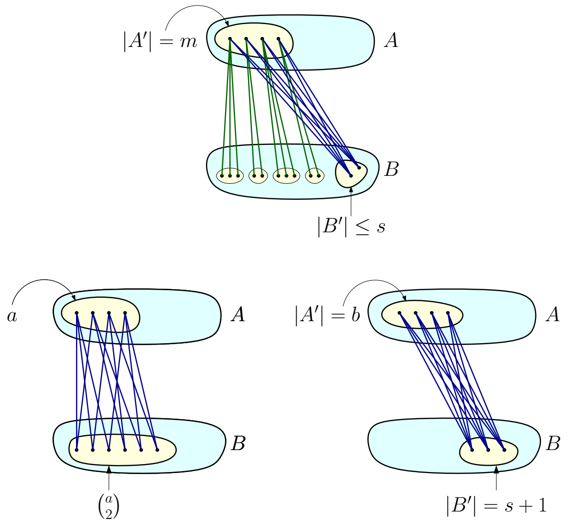
Lemma

Let $G = (A \cup B, E)$ be a bipartite graph.

If $|A| \geq \Theta(m,a,b,s)$ then at least one of the following holds:

- *A includes the principal vertices of a $K_a^{(1)}$;*
- *$\exists A' \subseteq A, B' \subseteq B$ s.t; every vertex in A' is adjacent to every vertex in B' and*
    - *either $|B'| = s+1$ and $|A'| = b$,*
    - *or $|B'| \leq s$, $|A'| = m$ and A' is 2-independent in $G - B'$.*

Finding 2-independent sets



Finding 2-independent sets

Lemma

Let $\mathcal{C}$ be a hereditary class.

- *Either* $\omega(\mathcal{C} \mathbin{\widetilde{\nabla}} \frac{1}{2}) = \infty$ *and*

$$\overline{\Phi}_{\mathcal{C}}(2) = \infty,$$

- *or* $\omega(\mathcal{C} \mathbin{\widetilde{\nabla}} \frac{1}{2}) < \infty$ *and*

$$\overline{\Phi}_{\mathcal{C}}(2) = \max\{s : \forall n \in \mathbb{N},\ K_{s,n} \in \mathcal{C}\}.$$

A $(2r+1)$-independent set in a $2r$-independent set

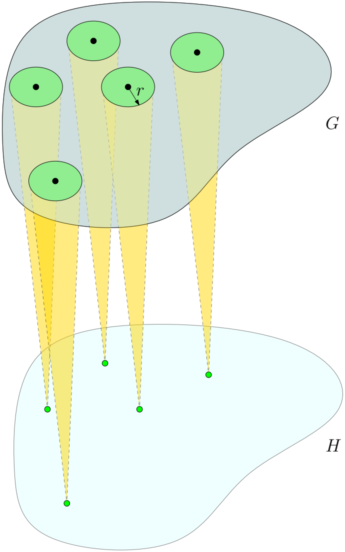
Lemma

Let A be a $2r$-independent subset of G of size at least $R(c,n)$. Then either $K_c \in G \triangledown r$ or A includes a $(2r+1)$-independent set of size n.

Proof.

Let $H \in G \triangledown r$ obtained by contracting the r-neighborhoods of vertices in A into A'. Either H contains a K_c or A' includes an independent set of size n of H, which corresponds to a $(2r+1)$-independent set of G included in A. □

A $(2r+1)$-independent set in a $2r$-independent set



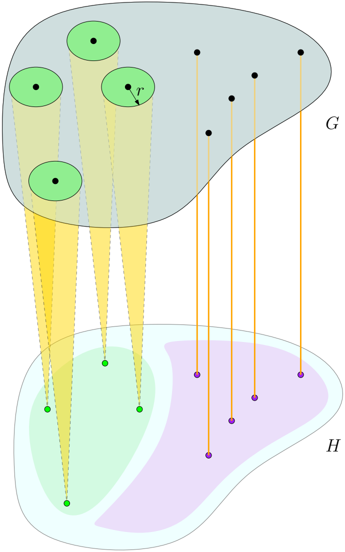
A $(2r+2)$-independent set in a $2r+1$-independent set

Lemma

Let A be a $(2r+1)$-independent set of G of size at least $\Theta(m, a, b, s)$. Then at least one of the following holds:

- $K_a \in G \,\nabla\, (r + 1/2)$ *(with centers in A);*
- $\exists A' \subseteq A, B' \subseteq B$ *such that* $B' \subseteq \bigcap_{x \in A'} N_{r+1}(x)$ *and*
  - *either* $|B'| = s + 1$ *and* $|A'| = b$,
  - *or* $|B'| \leq s, |A'| = m$ *and* A' *is* $(2r+2)$*-independent in* $G - B'$.

A $(2r+2)$-independent set in a $2r+1$-independent set



Characterizations

22 Wideness of a class

23 Finding d-independent sets

24 Characterizations

25 Applications

26 Homomorphism preservation theorems

Quasi wideness

Theorem

Let $\mathcal{C}$ be a hereditary class of graphs. The following are equivalent:

- *$\mathcal{C}$ is quasi-wide;*
- *$\mathcal{C}$ is uniformly quasi-wide;*
- *for every integer d, $\omega(\mathcal{C} \nabla d) < \infty$;*
- *for every integer d, $\omega(\mathcal{C} \mathbin{\widetilde{\nabla}} d) < \infty$;*
- *$\mathcal{C}$ is a class of nowhere dense graphs.*

extends the following:

Theorem (Dawar, Grohe, and Kreutzer, 2007)

If a class excludes locally a graph minor then it is quasi wide.

Almost wideness

Theorem

Let $\mathcal{C}$ be a hereditary class of graphs. The following are equivalent:

- *$\mathcal{C}$ is almost wide;*
- *$\mathcal{C}$ is uniformly almost wide;*
- *There are $s \in \mathbb{N}$ and $t : \mathbb{N} \to \mathbb{N}$ such that $K_{s,t(r)} \notin \mathcal{C} \triangledown r$ (for all $r \in \mathbb{N}$).*

extends the following:

Theorem (Atserias, Dawar, and Kolaitis, 2006)

If a class excludes a graph minor then it is almost wide.

Applications

22 Wideness of a class

23 Finding d-independent sets

24 Characterizations

25 Applications

26 Homomorphism preservation theorems

Forbidding induced paths

Theorem

There exists a function $F : \mathbb{N}^2 \to \mathbb{N}$ such that for every integers k, c and every graph G of order at least $F(k, c)$ one of the following condition holds:

- *either G includes P_{k+1} has an induced subgraph,*
- *or G includes K_c as a minor at depth $(k-1)/2$,*
- *or G has a non trivial involutive automorphism.*

Remark

$F(k, c)$ is really huge!!!

Proof.

Assume $P_{k+1} \not\subseteq_i G$ and $K_c \notin G \triangledown (k-1)/2$. Then

- As $K_c \notin G \triangledown (k-1)/2$, $\exists s = s(k-1, c)$, $\exists N = N(k-1, c)$ such that if $A \subseteq G$ and $|A| \geq N$ then $\exists S, |S| \leq s$ and $\alpha_{k-1}(G - S|A) \geq s + 2$;
- By contradiction, there exists no $P_N \subseteq G$ hence $\mathrm{td}(G) \leq N$;
- Thus if $|G| > X(N)$ then G has a non-trivial involutive automorphism.

$\square$

Powers of acyclic orientations

Let L be a linear order on $V(G)$ and let $\mathrm{Trans}_k(G)$ be the directed graph with $(x, y) \in E$ if $x <_L y$ and $\exists$ an L-increasing xy-path P of length at most k.

Theorem

For every $k, n \in \mathbb{N}$ there exists $N(k, n)$ such that for every graph G

$$\max_L \omega(\mathrm{Trans}_k(G)) > N(k,n) \implies \omega(G \,\widetilde{\nabla}\, \frac{k}{2}) > n$$

$$\omega(G \,\widetilde{\nabla}\, \frac{k}{2}) > n \implies \max_L \omega(\mathrm{Trans}_k(G)) > n$$

Proof

- Let $\omega = \omega(G \,\widetilde{\nabla}\, \frac{k}{2})$. $\exists s = s(k, \omega)$ s.t. $\forall A \subseteq V(G)$ with $|A| \geq C(k, \omega)$ $\exists S \subseteq V(G) : |S| \leq s$ and $\alpha_k(G - S|A) > 2^{s+1}$.
- Assume for contradiction that $\max_L \omega(\mathrm{Trans}_k(G)) \geq C(k, \omega)$.
  - Let A form a clique of size $C(k, \omega)$ in $\mathrm{Trans}_k(G)$. $\exists' S \; \exists A' \subseteq A$, $|S| \leq s$, $|A'| = 2^{s+1}$ and A' k-independent in $G - S$.
  - Let K be a clique of $\mathrm{Trans}_k(G)$; $\forall x$, $\geq \frac{|K|-1}{2}$ vertices of K are $< x$ or $> x$ hence form a clique in $\mathrm{Trans}_k(G - x)$.
  - By induction, as A' forms a clique in $\mathrm{Trans}_k(G)$ and $|A'| > 2^{s+1}$, $\exists a, a' \in A'$ in $\mathrm{Trans}_k(G - S)$ which are adjacent.

$\lightning$

Proof

- Let $\omega = \omega(G \,\widetilde{\nabla}\, \frac{k}{2})$. $\exists s = s(k,\omega)$ s.t. $\forall A \subseteq V(G)$ with $|A| \geq C(k,\omega)$ $\exists S \subseteq V(G) : |S| \leq s$ and $\alpha_k(G - S|A) > 2^{s+1}$.
- Assume for contradiction that $\max_L \omega(\mathrm{Trans}_k(G)) \geq C(k,\omega)$.
  - Let A form a clique of size $C(k,\omega)$ in $\mathrm{Trans}_k(G)$. $\exists' S\ \exists A' \subseteq A$, $|S| \leq s, |A'| = 2^{s+1}$ and A' k-independent in $G - S$.
  - Let K be a clique of $\mathrm{Trans}_k(G)$; $\forall x$, $\geq \frac{|K|-1}{2}$ vertices of K are $< x$ or $> x$ hence form a clique in $\mathrm{Trans}_k(G - x)$.
  - By induction, as A' forms a clique in $\mathrm{Trans}_k(G)$ and $|A'| > 2^{s+1}$, $\exists a, a' \in A'$ in $\mathrm{Trans}_k(G - S)$ which are adjacent.

$\lightning$

Proof

- Let $\omega = \omega(G \,\widetilde{\nabla}\, \frac{k}{2})$. $\exists s = s(k,\omega)$ s.t. $\forall A \subseteq V(G)$ with $|A| \geq C(k,\omega)$ $\exists S \subseteq V(G) : |S| \leq s$ and $\alpha_k(G - S|A) > 2^{s+1}$.
- Assume for contradiction that $\max_L \omega(\mathrm{Trans}_k(G)) \geq C(k,\omega)$.
  - Let A form a clique of size $C(k,\omega)$ in $\mathrm{Trans}_k(G)$. $\exists' S\ \exists A' \subseteq A$, $|S| \leq s, |A'| = 2^{s+1}$ and A' k-independent in $G - S$.
  - Let K be a clique of $\mathrm{Trans}_k(G)$; $\forall x$, $\geq \frac{|K|-1}{2}$ vertices of K are $< x$ or $> x$ hence form a clique in $\mathrm{Trans}_k(G - x)$.
  - By induction, as A' forms a clique in $\mathrm{Trans}_k(G)$ and $|A'| > 2^{s+1}$, $\exists a, a' \in A'$ in $\mathrm{Trans}_k(G - S)$ which are adjacent.

$\lightning$

Proof

- Let $\omega = \omega(G \,\widetilde{\triangledown}\, \frac{k}{2})$. $\exists s = s(k,\omega)$ s.t. $\forall A \subseteq V(G)$ with $|A| \geq C(k,\omega)$ $\exists S \subseteq V(G) : |S| \leq s$ and $\alpha_k(G - S|A) > 2^{s+1}$.
- Assume for contradiction that $\max_L \omega(\mathrm{Trans}_k(G)) \geq C(k,\omega)$.
  - Let A form a clique of size $C(k,\omega)$ in $\mathrm{Trans}_k(G)$. $\exists' S\ \exists A' \subseteq A$, $|S| \leq s, |A'| = 2^{s+1}$ and A' k-independent in $G - S$.
  - Let K be a clique of $\mathrm{Trans}_k(G)$; $\forall x$, $\geq \frac{|K|-1}{2}$ vertices of K are $< x$ or $> x$ hence form a clique in $\mathrm{Trans}_k(G - x)$.
  - By induction, as A' forms a clique in $\mathrm{Trans}_k(G)$ and $|A'| > 2^{s+1}$, $\exists a, a' \in A'$ in $\mathrm{Trans}_k(G - S)$ which are adjacent.

$\lightning$

Proof

- Let $\omega = \omega(G \,\widetilde{\nabla}\, \frac{k}{2})$. $\exists s = s(k, \omega)$ s.t. $\forall A \subseteq V(G)$ with $|A| \geq C(k, \omega)$ $\exists S \subseteq V(G) : |S| \leq s$ and $\alpha_k(G - S|A) > 2^{s+1}$.
- Assume for contradiction that $\max_L \omega(\mathrm{Trans}_k(G)) \geq C(k, \omega)$.
  - Let A form a clique of size $C(k, \omega)$ in $\mathrm{Trans}_k(G)$. $\exists' S$ $\exists A' \subseteq A$, $|S| \leq s$, $|A'| = 2^{s+1}$ and A' k-independent in $G - S$.
  - Let K be a clique of $\mathrm{Trans}_k(G)$; $\forall x$, $\geq \frac{|K|-1}{2}$ vertices of K are $< x$ or $> x$ hence form a clique in $\mathrm{Trans}_k(G - x)$.
  - By induction, as A' forms a clique in $\mathrm{Trans}_k(G)$ and $|A'| > 2^{s+1}$, $\exists a, a' \in A'$ in $\mathrm{Trans}_k(G - S)$ which are adjacent.

$\lightning$

Homomorphism preservation theorems

22 Wideness of a class

23 Finding d-independent sets

24 Characterizations

25 Applications

26 Homomorphism preservation theorems

Homomorphism preservation theorems

Theorem (Rossman, 2006)

Let φ be a first order formula.
Assume that for every **finite** $\mathbf{A}, \mathbf{B}$ *it holds*

$$\mathbf{A} \models \varphi \text{ and } \mathbf{A} \to \mathbf{B} \quad \Longrightarrow \quad \mathbf{B} \models \varphi.$$

Then φ is equivalent (in the **finite***) to an existential positive first order formula.*

Theorem

If the homomorphism preservation theorem holds for a hereditary class of graphs $\mathcal{C}$, it also holds for the class $\mathrm{Sub}_p(\mathcal{C})$ of all p-subdivisions of the graphs in $\mathcal{C}$.

Homomorphism preservation theorems

Theorem (Rossman, 2006)

Let φ be a first order formula.
Assume that for every **finite** $\mathbf{A}, \mathbf{B}$ *it holds*

$$\mathbf{A} \models \varphi \text{ and } \mathbf{A} \to \mathbf{B} \quad \Longrightarrow \quad \mathbf{B} \models \varphi.$$

Then φ is equivalent (in the **finite***) to an existential positive first order formula.*

Theorem

If the homomorphism preservation theorem holds for a hereditary class of graphs $\mathcal{C}$, it also holds for the class $\mathrm{Sub}_p(\mathcal{C})$ of all p-subdivisions of the graphs in $\mathcal{C}$.

Proof by functorial interpretation

Assume $\forall G, H \in \mathrm{Sub}_p(\mathcal{C}), \quad (G \models \Phi \ \wedge \ G \to H) \Rightarrow H \models \Phi.$

- By natural interpretation $I : \mathcal{C} \to \mathrm{Sub}_p(\mathcal{C})$, $\exists \Psi = I(\Phi)$ such that $\forall G \in \mathcal{C}, \quad G^{(p)} \models \Phi \iff G \models \Psi$;
- $(G \models \Psi \ \wedge G \to H) \Rightarrow (G^{(p)} \models \Phi \ \wedge G^{(p)} \to H^{(p)})$ $\Rightarrow H^{(p)} \models \Phi \Rightarrow H \models \Psi.$
- hence $\exists \mathcal{F} \subseteq \mathcal{C} : \ \forall G \in \mathcal{C}, \quad (\exists F \in \mathcal{F}, \ F \to G) \iff G \models \Psi$;
- in particular, $\forall F \in \mathcal{F}, F \models \Psi$ hence $F^{(p)} \models \Phi$; So, $\forall G \in \mathcal{C}$:
  - $(\exists F \in \mathcal{F}, F^{(p)} \to G^{(p)}) \Rightarrow G^{(p)} \models \Phi$;
  - $G^{(p)} \models \Phi \Rightarrow G \models \Psi \Rightarrow (\exists F \in \mathcal{F}, F \to G)$ $\Rightarrow (\exists F \in \mathcal{F}, F^{(p)} \to G^{(p)})$ $\square$

Proof by functorial interpretation

Assume $\forall G, H \in \mathrm{Sub}_p(\mathcal{C}), \quad (G \models \Phi \ \wedge \ G \to H) \Rightarrow H \models \Phi.$

- By natural interpretation $I : \mathcal{C} \to \mathrm{Sub}_p(\mathcal{C})$, $\exists \Psi = I(\Phi)$ such that $\forall G \in \mathcal{C}, \quad G^{(p)} \models \Phi \iff G \models \Psi$;
- $(G \models \Psi \ \wedge G \to H) \Rightarrow (G^{(p)} \models \Phi \ \wedge G^{(p)} \to H^{(p)})$ $\Rightarrow H^{(p)} \models \Phi \Rightarrow H \models \Psi$.
- hence $\exists \mathcal{F} \subseteq \mathcal{C} : \ \forall G \in \mathcal{C}, \quad (\exists F \in \mathcal{F}, \ F \to G) \iff G \models \Psi$;
- in particular, $\forall F \in \mathcal{F}, F \models \Psi$ hence $F^{(p)} \models \Phi$; So, $\forall G \in \mathcal{C}$:
  - $(\exists F \in \mathcal{F}, F^{(p)} \to G^{(p)}) \Rightarrow G^{(p)} \models \Phi$;
  - $G^{(p)} \models \Phi \Rightarrow G \models \Psi \Rightarrow (\exists F \in \mathcal{F}, F \to G)$
    $\Rightarrow (\exists F \in \mathcal{F}, F^{(p)} \to G^{(p)})$ $\square$

Proof by functorial interpretation

Assume $\forall G, H \in \mathrm{Sub}_p(\mathcal{C}), \quad (G \models \Phi \ \wedge \ G \to H) \Rightarrow H \models \Phi$.

- By natural interpretation $I : \mathcal{C} \to \mathrm{Sub}_p(\mathcal{C})$, $\exists \Psi = I(\Phi)$ such that $\forall G \in \mathcal{C}, \quad G^{(p)} \models \Phi \iff G \models \Psi$;
- $(G \models \Psi \ \wedge G \to H) \Rightarrow (G^{(p)} \models \Phi \ \wedge G^{(p)} \to H^{(p)})$ $\Rightarrow H^{(p)} \models \Phi \Rightarrow H \models \Psi$.
- hence $\exists \mathcal{F} \subseteq \mathcal{C} : \ \forall G \in \mathcal{C}, \quad (\exists F \in \mathcal{F}, \ F \to G) \iff G \models \Psi$;
- in particular, $\forall F \in \mathcal{F}, F \models \Psi$ hence $F^{(p)} \models \Phi$; So, $\forall G \in \mathcal{C}$:
  - $(\exists F \in \mathcal{F}, F^{(p)} \to G^{(p)}) \Rightarrow G^{(p)} \models \Phi$;
  - $G^{(p)} \models \Phi \Rightarrow G \models \Psi \Rightarrow (\exists F \in \mathcal{F}, F \to G)$ $\Rightarrow (\exists F \in \mathcal{F}, F^{(p)} \to G^{(p)})$ $\square$

Proof by functorial interpretation

Assume $\forall G, H \in \mathrm{Sub}_p(\mathcal{C}), \quad (G \models \Phi \ \wedge \ G \to H) \Rightarrow H \models \Phi$.

- By natural interpretation $I : \mathcal{C} \to \mathrm{Sub}_p(\mathcal{C})$, $\exists \Psi = I(\Phi)$ such that $\forall G \in \mathcal{C}, \quad G^{(p)} \models \Phi \iff G \models \Psi$;
- $(G \models \Psi \ \wedge G \to H) \Rightarrow (G^{(p)} \models \Phi \ \wedge G^{(p)} \to H^{(p)})$ $\Rightarrow H^{(p)} \models \Phi \Rightarrow H \models \Psi$.
- hence $\exists \mathcal{F} \subseteq \mathcal{C} : \ \forall G \in \mathcal{C}, \quad (\exists F \in \mathcal{F}, \ F \to G) \iff G \models \Psi$;
- in particular, $\forall F \in \mathcal{F}, F \models \Psi$ hence $F^{(p)} \models \Phi$; So, $\forall G \in \mathcal{C}$:
    - $(\exists F \in \mathcal{F}, F^{(p)} \to G^{(p)}) \Rightarrow G^{(p)} \models \Phi$;
    - $G^{(p)} \models \Phi \Rightarrow G \models \Psi \Rightarrow (\exists F \in \mathcal{F}, F \to G)$ $\Rightarrow (\exists F \in \mathcal{F}, F^{(p)} \to G^{(p)})$ □

Proof by functorial interpretation

Assume $\forall G, H \in \mathrm{Sub}_p(\mathcal{C})$, $\quad (G \models \Phi \ \wedge \ G \to H) \Rightarrow H \models \Phi$.

- By natural interpretation $I : \mathcal{C} \to \mathrm{Sub}_p(\mathcal{C})$, $\exists \Psi = I(\Phi)$ such that $\forall G \in \mathcal{C}$, $\quad G^{(p)} \models \Phi \iff G \models \Psi$;
- $(G \models \Psi \ \wedge G \to H) \Rightarrow (G^{(p)} \models \Phi \ \wedge G^{(p)} \to H^{(p)})$ $\Rightarrow H^{(p)} \models \Phi \Rightarrow H \models \Psi$.
- hence $\exists \mathcal{F} \subseteq \mathcal{C} :\ \forall G \in \mathcal{C}, \quad (\exists F \in \mathcal{F},\ F \to G) \iff G \models \Psi$;
- in particular, $\forall F \in \mathcal{F}, F \models \Psi$ hence $F^{(p)} \models \Phi$; So, $\forall G \in \mathcal{C}$:
  - $(\exists F \in \mathcal{F}, F^{(p)} \to G^{(p)}) \Rightarrow G^{(p)} \models \Phi$;
  - $G^{(p)} \models \Phi \Rightarrow G \models \Psi \Rightarrow (\exists F \in \mathcal{F}, F \to G)$ $\Rightarrow (\exists F \in \mathcal{F}, F^{(p)} \to G^{(p)})$ □

Proof by functorial interpretation

Assume $\forall G, H \in \mathrm{Sub}_p(\mathcal{C}), \quad (G \models \Phi \ \wedge \ G \to H) \Rightarrow H \models \Phi.$

- By natural interpretation $I : \mathcal{C} \to \mathrm{Sub}_p(\mathcal{C})$, $\exists \Psi = I(\Phi)$ such that $\forall G \in \mathcal{C}, \quad G^{(p)} \models \Phi \iff G \models \Psi$;
- $(G \models \Psi \ \wedge G \to H) \Rightarrow (G^{(p)} \models \Phi \ \wedge G^{(p)} \to H^{(p)})$
  $\Rightarrow H^{(p)} \models \Phi \Rightarrow H \models \Psi.$
- hence $\exists \mathcal{F} \subseteq \mathcal{C} : \ \forall G \in \mathcal{C}, \quad (\exists F \in \mathcal{F}, \ F \to G) \iff G \models \Psi$;
- in particular, $\forall F \in \mathcal{F}, F \models \Psi$ hence $F^{(p)} \models \Phi$; So, $\forall G \in \mathcal{C}$:
  - $(\exists F \in \mathcal{F}, F^{(p)} \to G^{(p)}) \Rightarrow G^{(p)} \models \Phi$;
  - $G^{(p)} \models \Phi \Rightarrow G \models \Psi \Rightarrow (\exists F \in \mathcal{F}, F \to G)$
    $\Rightarrow (\exists F \in \mathcal{F}, F^{(p)} \to G^{(p)})$ $\square$

Homomorphism preservation theorems

Theorem (Dawar, 2010)

Let φ be a first order formula and let $\mathcal{C}$ be a hereditary addable quasi wide class.
Assume that for every $\mathbf{A}, \mathbf{B} \in \mathcal{C}$ it holds

$$\mathbf{A} \models \varphi \text{ and } \mathbf{A} \to \mathbf{B} \quad \Longrightarrow \quad \mathbf{B} \models \varphi.$$

Then φ is equivalent (in $\mathcal{C}$) to an existential positive first order formula.

Corollary

The homomorphism preservation theorem holds for any hereditary addable nowhere dense class of graphs.

Homomorphism preservation theorems

Theorem (Dawar, 2010)

Let φ be a first order formula and let $\mathcal{C}$ be a hereditary addable quasi wide class.
Assume that for every $\mathbf{A}, \mathbf{B} \in \mathcal{C}$ it holds

$$\mathbf{A} \models \varphi \text{ and } \mathbf{A} \to \mathbf{B} \quad \Longrightarrow \quad \mathbf{B} \models \varphi.$$

Then φ is equivalent (in $\mathcal{C}$) to an existential positive first order formula.

Corollary

The homomorphism preservation theorem holds for any hereditary addable nowhere dense class of graphs.

Homomorphism preservation theorems

Corollary

For any monotone addable class of graphs $\mathcal{C}$ there exists an integer p such that the homomorphism preservation theorem holds for

$$\mathrm{Sub}_p(\mathcal{G}raph) \cap \mathcal{C},$$

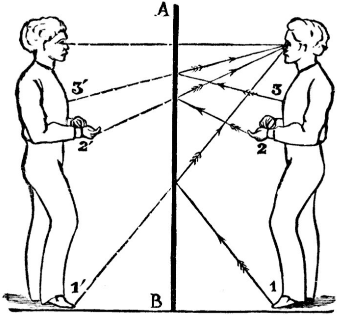
that is: to the subclass of p-subdivided graphs of $\mathcal{C}$.

Proof.

- if $\mathcal{C}$ is somewhere dense,
  $\exists p : \mathrm{Sub}_p(\mathcal{G}raph) \cap \mathcal{C} = \mathrm{Sub}_p(\mathcal{G}raph)$;
- if $\mathcal{C}$ is nowhere dense, true for any p.

$\square$

Homomorphism dualities



Homomorphisms and CSP

27 Homomorphisms and CSP

28 Finite dualities

29 Restricted dualities

30 Bounded expansion classes

31 Classes of structures

32 Classes of graphs closed by subdivisions

Homomorphisms

A *homomorphism* $G \to H$ is a mapping $f : V(G) \longrightarrow V(H)$ satisfying $\{x, y\} \in E(G) \implies \{f(x), f(y)\} \in E(H)$.

"edge preserving mappings"
(not only graphs; finite relational systems)

$f : G \to H \quad \equiv \quad H$-coloring

$G \to K_3 =$  $\equiv$ 3-coloring

Homomorphisms

A *homomorphism* $G \to H$ is a mapping $f : V(G) \longrightarrow V(H)$ satisfying $\quad \{x, y\} \in E(G) \quad \Longrightarrow \quad \{f(x), f(y)\} \in E(H).$

"edge preserving mappings"
(not only graphs; finite relational systems)

$$f : G \to H \qquad \equiv \qquad H\text{-coloring}$$

$G \to K_3 =$  $\equiv$ 3-coloring

Complexity

Theorem (Hell, Nešetřil, 1990)

H-coloring is hard $\iff$ H is non bipartite

Other proofs:

- Bulatov (graph theory and algebra)
- Siggers (combinatrics)
- Barto–Kozik (universal algebra)
- Kun–Szegedy (dynamical systems, PCP)

Complexity

Theorem (Hell, Nešetřil, 1990)

H-coloring is hard $\iff$ *H is non bipartite*

Other proofs:

- Bulatov (graph theory and algebra)
- Siggers (combinatorics)
- Barto–Kozik (universal algebra)
- Kun–Szegedy (dynamical systems, PCP)

All hard

Constraint Satisfaction Problems (CSP)

Problem: Assign values to variables so that all constraints are satisfied.

Examples

- SAT
- 3-COL
- $(x, y) \in \{(1, 1), (2, 3)\}$ and $(x, z, w) \in \{(2, 2, 1), (1, 3, 2), (2, 2, 2)\}$ ...

Theorem (Feder, Vardi)

Each constraint satisfaction problem is polynomially equivalent to $\rightarrow H$ for some digraph H.

Constraint Satisfaction Problems (CSP)

Problem: Assign values to variables so that all constraints are satisfied.

Examples

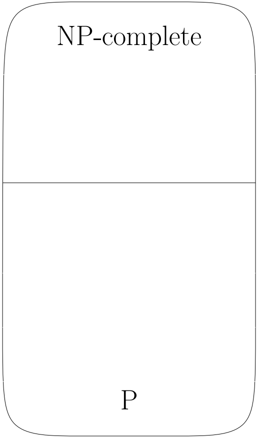
- SAT
- 3-COL
- $(x, y) \in \{(1,1),(2,3)\}$ and $(x, z, w) \in \{(2,2,1),(1,3,2),(2,2,2)\}$ ...

Theorem (Feder, Vardi)

Each constraint satisfaction problem is polynomially equivalent to $\to H$ for some digraph H.

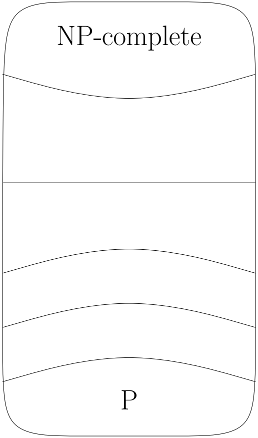
Dichotomy Conjecture

Problems in NP

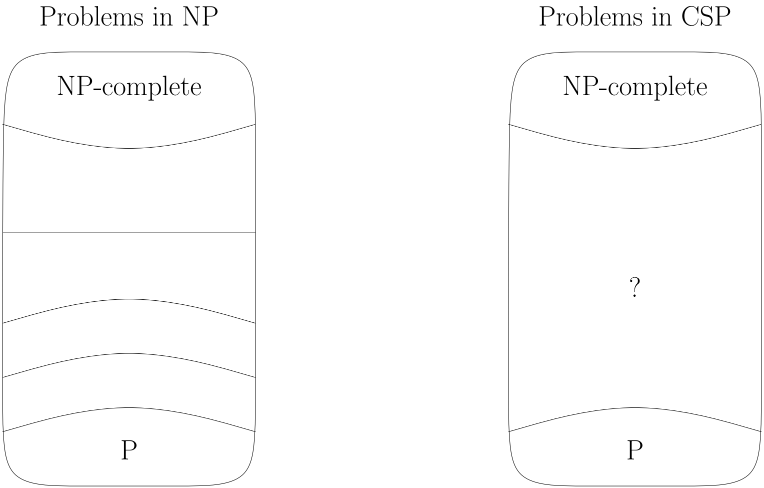


Dichotomy Conjecture

Problems in NP



Dichotomy Conjecture



Finite dualities

27 Homomorphisms and CSP

28 Finite dualities

29 Restricted dualities

30 Bounded expansion classes

31 Classes of structures

32 Classes of graphs closed by subdivisions

Critical graphs and duality

A graph G is *H-critical* if

$$
\begin{array}{lcr}
 & G \not\longrightarrow H & \\
\text{but} & G' \longrightarrow H & \text{for every } G' \subsetneq G
\end{array}
$$

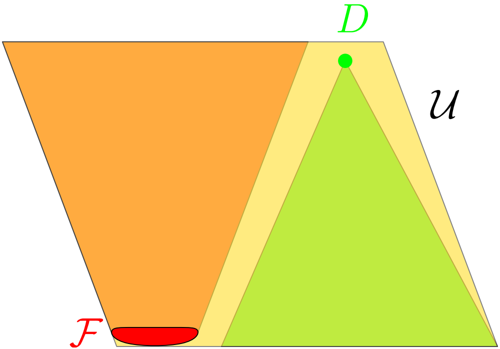
When are there finitely many H-critical graphs?
$\equiv$ do there exists $F_1, \ldots, F_t$ such that for every G

$$
\begin{array}{c}
F_1 \nrightarrow G \\
F_2 \nrightarrow G \\
\vdots \\
F_t \nrightarrow G
\end{array}
\iff G \to H
$$

Finite dualities

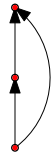
$$\mathrm{Forb}(\mathcal{F}) = \mathrm{CSP}(D)$$

$$(F \nrightarrow G \text{ for all } F \in \mathcal{F}) \iff G \to D$$

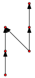
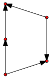


Finite dualities

Gallai, Hasse, Roy, Vitaver:

 $\not\longrightarrow G \iff G \longrightarrow$ 

Komárek:

 $\not\longrightarrow G \iff G \longrightarrow$ 

General:

$$F \not\longrightarrow G \iff G \longrightarrow D$$

Characterization of finite dualities

- Combinatorics
  - ($\mathcal{F}$ a set of trees) — Komárek; N., Tardif
  - (D^2 dismantable on the diagonal) — Larose, Lotten, Tardif
- Logic
  - (only FO definable CSP) — Atserias; Rossman
- Homomorphism poset
  - Gaps, Cuts and Bounds — (Heyting algebra)

Characterization of finite dualities

- Combinatorics
  - ($\mathcal{F}$ a set of trees) — Komárek; N., Tardif
  - (D^2 dismantable on the diagonal) — Larose, Lotten, Tardif
- Logic
  - (only FO definable CSP) — Atserias; Rossman
- Homomorphism poset
  - Gaps, Cuts and Bounds — (Heyting algebra)

Characterization of finite dualities

- Combinatorics
  - ($\mathcal{F}$ a set of trees) — Komárek; N., Tardif
  - (D^2 dismantable on the diagonal) — Larose, Lotten,Tardif
- Logic
  - (only FO definable CSP) — Atserias; Rossman
- Homomorphism poset
  - Gaps, Cuts and Bounds (Heyting algebra)

Characterization of finite dualities

- Combinatorics
  - ($\mathcal{F}$ a set of trees) — Komárek; N., Tardif
  - (D^2 dismantable on the diagonal) — Larose, Lotten, Tardif
- Logic
  - (only FO definable CSP) — Atserias; Rossman
- Homomorphism poset
  - Gaps, Cuts and Bounds — (Heyting algebra)

Characterization of finite dualities

- Combinatorics
  - ($\mathcal{F}$ a set of trees) — Komárek; N., Tardif
  - (D^2 dismantable on the diagonal) — Larose, Lotten, Tardif
- Logic
  - (only FO definable CSP) — Atserias; Rossman
- Homomorphism poset
  - Gaps, Cuts and Bounds (Heyting algebra)

Characterization of finite dualities

- Combinatorics
  - ($\mathcal{F}$ a set of trees) — Komárek; N., Tardif
  - (D^2 dismantable on the diagonal) — Larose, Lotten, Tardif
- Logic
  - (only FO definable CSP) — Atserias; Rossman
- Homomorphism poset
  - Gaps, Cuts and Bounds (Heyting algebra)

Characterization of finite dualities

- Combinatorics
  - ($\mathcal{F}$ a set of trees) — Komárek; N., Tardif
  - (D^2 dismantable on the diagonal) — Larose, Lotten, Tardif
- Logic
  - (only FO definable CSP) — Atserias; Rossman
- Homomorphism poset
  - Gaps, Cuts and Bounds — (Heyting algebra)

Restricted dualities

27 Homomorphisms and CSP

28 Finite dualities

29 Restricted dualities

30 Bounded expansion classes

31 Classes of structures

32 Classes of graphs closed by subdivisions

Restricted dualities

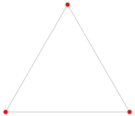
$\mathcal{C}$-restricted duality

$$\mathrm{Forb}(\mathcal{F}) \cap \mathcal{C} = \mathrm{CSP}(D) \cap \mathcal{C} \quad \text{and} \quad \boxed{D \in \mathrm{Forb}(\mathcal{F})}$$

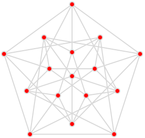
$$\equiv$$

$$\begin{cases} \forall G \in \mathcal{C}: & (\forall F \in \mathcal{F},\ F \not\longrightarrow G) \iff (G \longrightarrow D) \\ \forall F \in \mathcal{F}: & F \not\longrightarrow D. \end{cases}$$

Example: $\forall$ planar G,



$\not\longrightarrow \ G \quad \iff \quad G \ \longrightarrow$



Restricted dualities

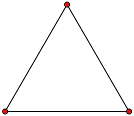
$\mathcal{C}$-restricted duality

$$\mathrm{Forb}(\mathcal{F}) \cap \mathcal{C} = \mathrm{CSP}(D) \cap \mathcal{C} \quad \text{and} \quad \boxed{D \in \mathrm{Forb}(\mathcal{F})}$$

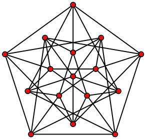
$$\equiv$$

$$\begin{cases} \forall G \in \mathcal{C}: & (\forall F \in \mathcal{F},\ F \not\longrightarrow G) \iff (G \longrightarrow D) \\ \forall F \in \mathcal{F}: & F \not\longrightarrow D. \end{cases}$$

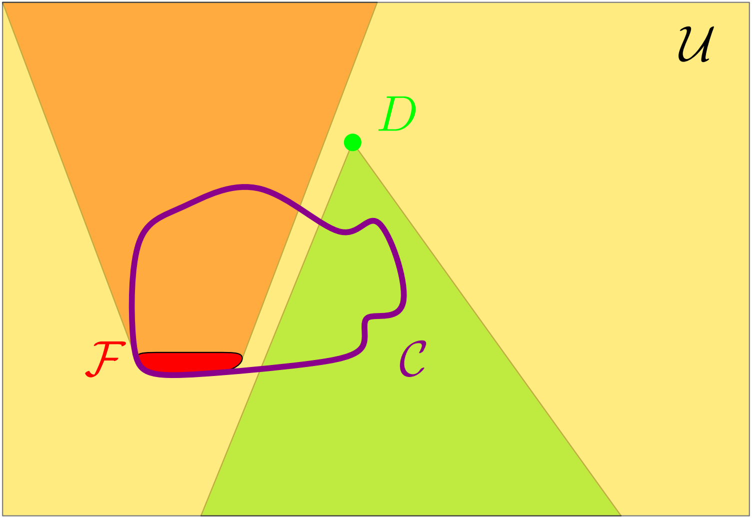
Example: $\forall$ planar G,



$\not\longrightarrow G \iff G \longrightarrow$



Restricted duality



All Restricted Dualities

$\mathcal{C}$ has All Restricted Dualities (ARD)

iff

for every finite set $\mathcal{F} \subseteq \mathcal{C}$ (of connected structures) there exists $D_{\mathcal{F}} \in \mathrm{Forb}(\mathcal{F})$ such that

$$\mathcal{C} \cap \mathrm{Forb}(\mathcal{F}) = \mathcal{C} \cap \mathrm{CSP}(D_{\mathcal{F}})$$

- Bounded degree graphs have all restricted dualities (ARD) (Häggkvist, Hell)
- Planar graphs have ARD

All Restricted Dualities

$\mathcal{C}$ has All Restricted Dualities (ARD)

iff

for every finite set $\mathcal{F} \subseteq \mathcal{C}$ (of connected structures) there exists $D_{\mathcal{F}} \in \mathrm{Forb}(\mathcal{F})$ such that

$$\mathcal{C} \cap \mathrm{Forb}(\mathcal{F}) = \mathcal{C} \cap \mathrm{CSP}(D_{\mathcal{F}})$$

- Bounded degree graphs have all restricted dualities (ARD) (Hăggkvist, Hell)
- Planar graphs have ARD

Characterization by metric properties

$$\mathrm{dist}_L(\mathbf{A},\mathbf{B}) = 2^{-k}$$
$$k = \min\{|\mathbf{C}| : (\mathbf{C} \to \mathbf{A} \ \wedge\ \mathbf{C} \nrightarrow \mathbf{B}) \text{ or } (\mathbf{C} \nrightarrow \mathbf{A} \ \wedge\ \mathbf{C} \to \mathbf{B})\}$$

For $\epsilon > 0$ let $\phi^\epsilon(\mathbf{A}) = \min\{|\mathbf{B}| : \mathbf{A} \to \mathbf{B} \text{ and } \mathrm{dist}_L(\mathbf{A},\mathbf{B}) < \epsilon\}$.

Theorem

For a class $\mathcal{C}$ the following conditions are equivalent:

1. $\mathcal{C}$ has all restricted dualities,
2. for every $\epsilon > 0$, $\sup_{\mathbf{A}\in\mathcal{C}} \phi^\epsilon(\mathbf{A}) < \infty$.

Characterization by metric properties

$$\mathrm{dist}_L(\mathbf{A},\mathbf{B}) = 2^{-k}$$
$$k = \min\{|\mathbf{C}| : (\mathbf{C} \to \mathbf{A} \ \wedge \ \mathbf{C} \nrightarrow \mathbf{B}) \text{ or } (\mathbf{C} \nrightarrow \mathbf{A} \ \wedge \ \mathbf{C} \to \mathbf{B})\}$$

For $\epsilon > 0$ let $\phi^\epsilon(\mathbf{A}) = \min\{|\mathbf{B}| : \mathbf{A} \to \mathbf{B} \text{ and } \mathrm{dist}_L(\mathbf{A},\mathbf{B}) < \epsilon\}$.

Theorem

For a class $\mathcal{C}$ the following conditions are equivalent:

1. $\mathcal{C}$ has all restricted dualities,
2. for every $\epsilon > 0$, $\sup_{\mathbf{A}\in\mathcal{C}} \phi^\epsilon(\mathbf{A}) < \infty$.

Proof $\Longrightarrow$

- Assume $\mathcal{C}$ has ARD, let $\epsilon > 0$ and $t \geq -\log_2 \epsilon$.

$$\begin{aligned} \mathcal{F}_t(\mathbf{A}) &= \{\mathbf{T} \text{ connected core : } |\mathbf{T}| \leq t \text{ and } \mathbf{T} \nrightarrow \mathbf{A}\}, \\ \mathbf{D}_\mathbf{T} &= \text{ dual of } \mathbf{T} \in \mathcal{F}_t(\mathbf{A}) \text{ wrt } \mathcal{C}, \\ \mathbf{A}' &= \prod_{\mathbf{T} \in \mathcal{F}_t(\mathbf{A})} \mathbf{D}_\mathbf{T}. \end{aligned}$$

- $\forall \mathbf{T} \in \mathcal{F}_t(\mathbf{A})$, $\mathbf{T} \nrightarrow \mathbf{A}$ hence $\mathbf{A} \to \mathbf{D}_\mathbf{T}$. Thus $\mathbf{A} \to \mathbf{A}'$.
- Let $\mathbf{T}'$ be connected, $|\mathbf{T}'| \leq t$.

$$\begin{aligned} &\mathbf{T}' \to \mathbf{A} \Longrightarrow \mathbf{T}' \to \mathbf{A}' \\ &\mathbf{T}' \nrightarrow \mathbf{A} \Longrightarrow \mathrm{Core}(\mathbf{T}') \in \mathcal{F}_t(\mathbf{A}) \Longrightarrow \mathbf{A}' \to \mathbf{D}_{\mathbf{T}'} \Longrightarrow \mathbf{T}' \nrightarrow \mathbf{A}' \end{aligned}$$

Thus $\mathrm{dist}_L(\mathbf{A}, \mathbf{A}') \leq \epsilon$ and $|\phi^\epsilon(\mathbf{A})| \leq |\mathbf{A}'| \leq C_\epsilon$.

Proof $\Longrightarrow$

- Assume $\mathcal{C}$ has ARD, let $\epsilon > 0$ and $t \geq -\log_2 \epsilon$.

$$\begin{aligned}
\mathcal{F}_t(\mathbf{A}) &= \{\mathbf{T} \text{ connected core}: \ |\mathbf{T}| \leq t \text{ and } \mathbf{T} \nrightarrow \mathbf{A}\},\\
\mathbf{D}_{\mathbf{T}} &= \text{ dual of } \mathbf{T} \in \mathcal{F}_t(\mathbf{A}) \text{ wrt } \mathcal{C},\\
\mathbf{A}' &= \prod_{\mathbf{T} \in \mathcal{F}_t(\mathbf{A})} \mathbf{D}_{\mathbf{T}}.
\end{aligned}$$

- $\forall \mathbf{T} \in \mathcal{F}_t(\mathbf{A})$, $\mathbf{T} \nrightarrow \mathbf{A}$ hence $\mathbf{A} \to \mathbf{D}_{\mathbf{T}}$. Thus $\mathbf{A} \to \mathbf{A}'$.
- Let $\mathbf{T}'$ be connected, $|\mathbf{T}'| \leq t$.

$$\begin{aligned}
&\mathbf{T}' \to \mathbf{A} \Longrightarrow \mathbf{T}' \to \mathbf{A}'\\
&\mathbf{T}' \nrightarrow \mathbf{A} \Longrightarrow \mathrm{Core}(\mathbf{T}') \in \mathcal{F}_t(\mathbf{A}) \Longrightarrow \mathbf{A}' \to \mathbf{D}_{\mathbf{T}'} \Longrightarrow \mathbf{T}' \nrightarrow \mathbf{A}'
\end{aligned}$$

Thus $\mathrm{dist}_L(\mathbf{A}, \mathbf{A}') \leq \epsilon$ and $|\phi^\epsilon(\mathbf{A})| \leq |\mathbf{A}'| \leq C_\epsilon$.

Proof $\Longleftarrow$

- Assume $\sup_{\mathbf{A}\in\mathcal{C}} |\phi^\epsilon(\mathbf{A})| < \infty$ for every $\epsilon > 0$.
  Let $\mathbf{F}$ be connected, $t \geq |\mathbf{F}|, \epsilon = 2^{-t}$,

$$\mathcal{D} = \{\phi^\epsilon(\mathbf{A}) : \ \mathbf{A} \in \mathcal{C} \ \wedge \ \mathbf{F} \nrightarrow \mathbf{A}\} \qquad \text{(finite)}$$

$$D_t(\mathbf{F}) = \coprod_{\mathbf{X}\in\mathcal{D}} \mathbf{X}.$$

  - $\mathbf{F} \nrightarrow D_t(\mathbf{F})$
    (for otherwise $\exists \phi^\epsilon(\mathbf{B}) \in \mathcal{D}$: $\mathbf{F} \longrightarrow \phi^\epsilon(\mathbf{B}) \wedge \mathbf{F} \nrightarrow \mathbf{B}$,
    contradicts $\mathrm{dist}_L(\phi^\epsilon(\mathbf{B}), \mathbf{B}) \leq 2^{-|\mathbf{F}|}$).
  - $\mathbf{F} \rightarrow \mathbf{A} \Longrightarrow \mathbf{A} \nrightarrow D_t(\mathbf{F})$ (for otherwise $\mathbf{F} \rightarrow D_t(\mathbf{F})$)
  - $\mathbf{F} \nrightarrow \mathbf{A} \Longrightarrow \phi^\epsilon(\mathbf{A}) \in \mathcal{D} \Longrightarrow \mathbf{A} \rightarrow D_t(\mathbf{F})$.

Proof $\Longleftarrow$

- Assume $\sup_{\mathbf{A}\in\mathcal{C}} |\phi^\epsilon(\mathbf{A})| < \infty$ for every $\epsilon > 0$.
  Let $\mathbf{F}$ be connected, $t \geq |\mathbf{F}|, \epsilon = 2^{-t}$,

$$\mathcal{D} = \{\phi^\epsilon(\mathbf{A}) :\ \mathbf{A} \in \mathcal{C}\ \wedge\ \mathbf{F} \nrightarrow \mathbf{A}\} \qquad \text{(finite)}$$

$$D_t(\mathbf{F}) = \coprod_{\mathbf{X}\in\mathcal{D}} \mathbf{X}.$$

  - $\mathbf{F} \nrightarrow D_t(\mathbf{F})$
    (for otherwise $\exists\phi^\epsilon(\mathbf{B}) \in \mathcal{D} :\ \mathbf{F} \longrightarrow \phi^\epsilon(\mathbf{B}) \wedge \mathbf{F} \nrightarrow \mathbf{B}$,
    contradicts $\mathrm{dist}_L(\phi^\epsilon(\mathbf{B}), \mathbf{B}) \leq 2^{-|\mathbf{F}|}$).
  - $\mathbf{F} \to \mathbf{A} \Longrightarrow \mathbf{A} \nrightarrow D_t(\mathbf{F})$ (for otherwise $\mathbf{F} \to D_t(\mathbf{F})$)
  - $\mathbf{F} \nrightarrow \mathbf{A} \Longrightarrow \phi^\epsilon(\mathbf{A}) \in \mathcal{D} \Longrightarrow \mathbf{A} \to D_t(\mathbf{F})$.

Proof $\Longleftarrow$

- Assume $\sup_{\mathbf{A}\in\mathcal{C}} |\phi^\epsilon(\mathbf{A})| < \infty$ for every $\epsilon > 0$.
  Let $\mathbf{F}$ be connected, $t \geq |\mathbf{F}|, \epsilon = 2^{-t}$,

$$\mathcal{D} = \{\phi^\epsilon(\mathbf{A}) :\ \mathbf{A} \in \mathcal{C}\ \wedge\ \mathbf{F} \nrightarrow \mathbf{A}\} \qquad \text{(finite)}$$

$$D_t(\mathbf{F}) = \coprod_{\mathbf{X}\in\mathcal{D}} \mathbf{X}.$$

  - $\mathbf{F} \nrightarrow D_t(\mathbf{F})$
    (for otherwise $\exists \phi^\epsilon(\mathbf{B}) \in \mathcal{D}:\ \mathbf{F} \longrightarrow \phi^\epsilon(\mathbf{B}) \wedge \mathbf{F} \nrightarrow \mathbf{B}$,
    contradicts $\mathrm{dist}_L(\phi^\epsilon(\mathbf{B}), \mathbf{B}) \leq 2^{-|\mathbf{F}|}$).
  - $\mathbf{F} \to \mathbf{A} \Longrightarrow \mathbf{A} \nrightarrow D_t(\mathbf{F})$ (for otherwise $\mathbf{F} \to D_t(\mathbf{F})$)
  - $\mathbf{F} \nrightarrow \mathbf{A} \Longrightarrow \phi^\epsilon(\mathbf{A}) \in \mathcal{D} \Longrightarrow \mathbf{A} \to D_t(\mathbf{F})$.

Bounded expansion classes

27 Homomorphisms and CSP

28 Finite dualities

29 Restricted dualities

30 Bounded expansion classes

31 Classes of structures

32 Classes of graphs closed by subdivisions

Bounded expansion classes

Definition

A class $\mathcal{C}$ of graphs has *bounded expansion* if

$$\forall i \in \mathbb{N}, \quad \sup_{G \in \mathcal{C} \,\nabla\, i} \frac{\|G\|}{|G|} < \infty.$$

Equivalent statements

$$\forall i, \quad \sup_{G \in \mathcal{C} \,\widetilde{\nabla}\, i} \frac{\|G\|}{|G|} < \infty \qquad \forall p, \quad \sup_{G \in \mathcal{C}} \chi_p(G) < \infty$$

$$\forall i, \quad \sup_{G \in \mathcal{C} \,\nabla\, i} \chi(G) < \infty \qquad \forall i, \quad \sup_{G \in \mathcal{C} \,\widetilde{\nabla}\, i} \chi(G) < \infty$$

Bounded expansion classes

Definition

A class $\mathcal{C}$ of graphs has *bounded expansion* if

$$\forall i \in \mathbb{N}, \qquad \sup_{G \in \mathcal{C} \,\nabla\, i} \frac{\|G\|}{|G|} < \infty.$$

Equivalent statements

$$\forall i, \qquad \sup_{G \in \mathcal{C} \,\widetilde{\nabla}\, i} \frac{\|G\|}{|G|} < \infty \qquad\qquad \forall p, \qquad \sup_{G \in \mathcal{C}} \chi_p(G) < \infty$$

$$\forall i, \qquad \sup_{G \in \mathcal{C} \,\nabla\, i} \chi(G) < \infty \qquad\qquad \forall i, \qquad \sup_{G \in \mathcal{C} \,\widetilde{\nabla}\, i} \chi(G) < \infty$$

Bounded expansion classes

Definition

A class $\mathcal{C}$ of relational structures has *bounded expansion* if the class $\mathrm{Gaifman}(\mathcal{C})$ has bounded expansion.

Lemma

A class $\mathcal{C}$ of relational structures has bounded expansion if and only if the class $\mathrm{Incid}(\mathcal{C})$ has bounded expansion.

Bounded expansion classes

Definition

A class $\mathcal{C}$ of relational structures has *bounded expansion* if the class $\mathrm{Gaifman}(\mathcal{C})$ has bounded expansion.

Lemma

A class $\mathcal{C}$ of relational structures has bounded expansion if and only if the class $\mathrm{Incid}(\mathcal{C})$ has bounded expansion.

Bounded expansion classes

Proof.

Let p be the maximum arity of a symbol.

$$\mathrm{Incid}(\mathcal{C}) \subseteq (\mathrm{Gaifman}(\mathcal{C}) \bullet K_2) \,\triangledown\, 0,$$
$$\mathrm{Gaifman}(\mathcal{C}) \subseteq (\mathrm{Incid}(\mathcal{C}) \bullet K_{\binom{p}{2}}) \,\widetilde{\triangledown}\, \frac{1}{2}.$$

Thus

$$\begin{aligned}\mathrm{Gaifman}(\mathcal{C}) \text{ has BE} &\Longrightarrow (\mathrm{Gaifman}(\mathcal{C}) \bullet K_2) \,\triangledown\, 0 \text{ has BE} \\ &\Longrightarrow \mathrm{Incid}(\mathcal{C}) \text{ has BE} \\ \mathrm{Incid}(\mathcal{C}) \text{ has BE} &\Longrightarrow (\mathrm{Incid}(\mathcal{C}) \bullet K_{\binom{p}{2}}) \,\widetilde{\triangledown}\, \frac{1}{2} \text{ has BE} \\ &\Longrightarrow \mathrm{Gaifman}(\mathcal{C}) \text{ has BE}\end{aligned}$$

$\square$

Approximation of a structure

Construction

Let $p \in \mathbb{N}$. Let γ be a χ_p-coloring of $\mathrm{Gaifman}(\mathbf{A})$ with N colors. For $I \in \binom{[N]}{p}$, let $\mathbf{A}_I = \mathrm{Core}(\mathbf{A}[\gamma^{-1}(I)])$.

Define $F_p(\mathbf{A})$:

- base set $W_1 \cup \cdots \cup W_N$ where

$$W_c = \{\zeta : I \in \binom{N}{p} \text{ with } c \in I \ \longrightarrow \mathbf{A}_I\}$$

- relations $\{(\zeta_1, \ldots, \zeta_r) \in W_{c_1} \times \cdots \times W_{c_r}\}$ such that

$$(\zeta_1(I), \ldots, \zeta_r(I)) \text{ relation of } \mathbf{A}_I \quad \forall I \supseteq \{c_1, \ldots, c_r\}$$

Approximation of a structure

Properties

- $\mathbf{A} \longrightarrow F_p(\mathbf{A})$,
- $\forall \mathbf{T}, |\mathbf{T}| \leq p : \qquad (\mathbf{T} \to \mathbf{A}) \iff (\mathbf{T} \to F_p(\mathbf{A}))$,
- There exists C_p depending only on p and the signature of $\mathbf{A}$ such that $|F_p(\mathbf{A})| \leq N\, C_p^{\binom{N}{p-1}}$, where $N = \chi_p(\mathrm{Gaifman}(\mathbf{A}))$.

Hence $|\phi^{2^{-p}}(\mathbf{A})| \leq f(p, \chi_p(\mathrm{Gaifman}(\mathbf{A})))$.

Remark

Actually F_p is the the right-adjoint of the functor $\mathbf{A} \mapsto (\mathbf{A}_I)_{I \in \binom{[N]}{p}}$.

Bounded expansion classes and restricted dualities

Theorem

Bounded expansion classes have all restricted dualities.

Precise version

Let $p \in \mathbb{N}$. If $\chi_p(\mathrm{Gaifman}(\mathcal{C})) < \infty$ then every connected $\mathbf{F} \in \mathcal{C}$ has a restricted dual (wrt $\mathcal{C}$).

Problem

Are there other classes which have all restricted dualities?

Bounded expansion classes and restricted dualities

Theorem

Bounded expansion classes have all restricted dualities.

Precise version

Let $p \in \mathbb{N}$. If $\chi_p(\mathrm{Gaifman}(\mathcal{C})) < \infty$ then every connected $\mathbf{F} \in \mathcal{C}$ has a restricted dual (wrt $\mathcal{C}$).

Problem

Are there other classes which have all restricted dualities?

Classes of structures

27 Homomorphisms and CSP

28 Finite dualities

29 Restricted dualities

30 Bounded expansion classes

31 Classes of structures

32 Classes of graphs closed by subdivisions

Reorientation of structures

Let **A** be a structure. A structure **A**′ with the same base set is a *reorientation* of **A** if

$$(x_1, \ldots, x_r) \in R^{\mathbf{B}} \implies \exists \sigma \in \mathfrak{S}_r : (x_{\sigma(1)}, \ldots, x_{\sigma(r)}) \in R^{\mathbf{A}}$$
$$(x_1, \ldots, x_r) \in R^{\mathbf{A}} \implies \exists \sigma \in \mathfrak{S}_r : (x_{\sigma(1)}, \ldots, x_{\sigma(r)}) \in R^{\mathbf{B}}$$

Circuits of **A** $\equiv$ circuits of $\mathrm{Incid}(\mathbf{A})$.

All possible reorientations: $\mathcal{C} \longmapsto \mathcal{C}_{\mathrm{orient}}$

Acyclic structures

Let **A** be a structure and let $<$ be a linear order on its base set. $\mathbf{A}_<$ is the structure with base set A where

$$(x_1, \dots, x_r) \in R^{\mathbf{A}_<}$$

$$\iff$$

$$x_1 < \cdots < x_r \quad \wedge \quad \exists \sigma \in \mathfrak{S}_r : \ (x_{\sigma(1)}, \dots, x_{\sigma(r)}) \in R^{\mathbf{A}}$$

All possible linear orderings $\equiv$ all possible acyclic reorientations

$$\mathcal{C} \quad \longmapsto \quad \mathcal{C}_{\mathrm{acyc}}$$

Classes with all restricted dualities

Theorem

Let $\mathcal{C}$ be a class of structures. The following are equivalent:

1. *$\mathcal{C}$ has bounded expansion,*
2. *$\mathcal{C}_{\mathrm{orient}}$ has all restricted dualities,*
3. *for every integer p, there exists a structure $\mathbf{D}_p$ without a circuit of length at most p such that*

$$\forall \mathbf{A} \in \mathcal{C}_{\mathrm{acyc}}, \qquad \mathbf{A} \to \mathbf{D}_p.$$

Proof

- (1) $\Rightarrow$ (2) already proved; (2) $\Rightarrow$ (3) is straightforward.
- (3) $\Rightarrow$ (1) by contradiction.

$$\neg(1) \Rightarrow \mathrm{Incid}(\mathcal{C}) \text{ does not have bounded expansion}$$
$$\Rightarrow \exists p : \chi(\mathrm{Incid}(\mathcal{C}) \,\widetilde{\nabla}\, p) = \infty.$$

  - $\exists H, S, \mathbf{A}$: $\chi(H) > |\mathbf{D}_{p+1}|$, $\delta(H) >$ maximum arity, S is a $\leq p$-subdivision of H and $S \subseteq \mathrm{Incid}(\mathbf{A}) \in \mathrm{Incid}(\mathcal{C})$;
  - the branching vertices of S correspond to points of $\mathbf{A}$;
  - consider $<$ such that every branch of S is monotone;
  - (3) $\Rightarrow \exists f : \mathbf{A}_< \to \mathbf{D}_{p+1}$;
  - two endpoints of a branch of S have distinct images by f;
  - $\Rightarrow \chi(H) \leq |\mathbf{D}_{p+1}|$.

$\lightning$

Proof

- (1) $\Rightarrow$ (2) already proved; (2) $\Rightarrow$ (3) is straightforward.
- (3) $\Rightarrow$ (1) by contradiction.

$\neg(1) \Rightarrow \mathrm{Incid}(\mathcal{C})$ does not have bounded expansion
$\Rightarrow \exists p : \chi(\mathrm{Incid}(\mathcal{C}) \mathbin{\widetilde{\nabla}} p) = \infty.$

- $\exists H, S, \mathbf{A}$: $\chi(H) > |\mathbf{D}_{p+1}|$, $\delta(H) >$ maximum arity, S is a $\leq p$-subdivision of H and $S \subseteq \mathrm{Incid}(\mathbf{A}) \in \mathrm{Incid}(\mathcal{C})$;
- the branching vertices of S correspond to points of $\mathbf{A}$;
- consider $<$ such that every branch of S is monotone;
- (3) $\Rightarrow \exists f : \mathbf{A}_< \to \mathbf{D}_{p+1}$;
- two endpoints of a branch of S have distinct images by f;
- $\Rightarrow \chi(H) \leq |\mathbf{D}_{p+1}|$.

$\lightning$

Proof

- (1) $\Rightarrow$ (2) already proved; (2) $\Rightarrow$ (3) is straightforward.
- (3) $\Rightarrow$ (1) by contradiction.

$\neg(1) \Rightarrow$ $\mathrm{Incid}(\mathcal{C})$ does not have bounded expansion

$\Rightarrow$ $\exists p : \chi(\mathrm{Incid}(\mathcal{C}) \,\widetilde{\nabla}\, p) = \infty$.

- $\exists H, S, \mathbf{A}$: $\chi(H) > |\mathbf{D}_{p+1}|$, $\delta(H) >$ maximum arity, S is a $\leq p$-subdivision of H and $S \subseteq \mathrm{Incid}(\mathbf{A}) \in \mathrm{Incid}(\mathcal{C})$;
- the branching vertices of S correspond to points of $\mathbf{A}$;
- consider $<$ such that every branch of S is monotone;
- (3) $\Rightarrow \exists f : \mathbf{A}_< \to \mathbf{D}_{p+1}$;
- two endpoints of a branch of S have distinct images by f;
- $\Rightarrow \chi(H) \leq |\mathbf{D}_{p+1}|$.

$\lightning$

Proof

- (1) $\Rightarrow$ (2) already proved; (2) $\Rightarrow$ (3) is straightforward.
- (3) $\Rightarrow$ (1) by contradiction.

$\neg(1) \Rightarrow$ $\mathrm{Incid}(\mathcal{C})$ does not have bounded expansion
$\Rightarrow$ $\exists p : \chi(\mathrm{Incid}(\mathcal{C}) \,\widetilde{\nabla}\, p) = \infty$.

- $\exists H, S, \mathbf{A}$: $\chi(H) > |\mathbf{D}_{p+1}|$, $\delta(H) >$ maximum arity, S is a $\leq p$-subdivision of H and $S \subseteq \mathrm{Incid}(\mathbf{A}) \in \mathrm{Incid}(\mathcal{C})$;
- the branching vertices of S correspond to points of $\mathbf{A}$;
- consider $<$ such that every branch of S is monotone;
- (3) $\Rightarrow \exists f : \mathbf{A}_< \to \mathbf{D}_{p+1}$;
- two endpoints of a branch of S have distinct images by f;
- $\Rightarrow \chi(H) \leq |\mathbf{D}_{p+1}|$.

$\lightning$

Proof

- (1) $\Rightarrow$ (2) already proved; (2) $\Rightarrow$ (3) is straightforward.
- (3) $\Rightarrow$ (1) by contradiction.

$\neg(1) \Rightarrow \mathrm{Incid}(\mathcal{C})$ does not have bounded expansion
$\Rightarrow \exists p : \chi(\mathrm{Incid}(\mathcal{C}) \,\widetilde{\nabla}\, p) = \infty$.

- $\exists H, S, \mathbf{A}$: $\chi(H) > |\mathbf{D}_{p+1}|$, $\delta(H) >$ maximum arity, S is a $\leq p$-subdivision of H and $S \subseteq \mathrm{Incid}(\mathbf{A}) \in \mathrm{Incid}(\mathcal{C})$;
- the branching vertices of S correspond to points of $\mathbf{A}$;
- consider $<$ such that every branch of S is monotone;
- (3) $\Rightarrow \exists f : \mathbf{A}_< \to \mathbf{D}_{p+1}$;
- two endpoints of a branch of S have distinct images by f;
- $\Rightarrow \chi(H) \leq |\mathbf{D}_{p+1}|$.

↯

Proof

- (1) $\Rightarrow$ (2) already proved; (2) $\Rightarrow$ (3) is straightforward.
- (3) $\Rightarrow$ (1) by contradiction.

$\neg(1) \Rightarrow \mathrm{Incid}(\mathcal{C})$ does not have bounded expansion

$\Rightarrow \exists p : \chi(\mathrm{Incid}(\mathcal{C}) \,\widetilde{\nabla}\, p) = \infty.$

- $\exists H, S, \mathbf{A}$: $\chi(H) > |\mathbf{D}_{p+1}|$, $\delta(H) >$ maximum arity, S is a $\leq p$-subdivision of H and $S \subseteq \mathrm{Incid}(\mathbf{A}) \in \mathrm{Incid}(\mathcal{C})$;
- the branching vertices of S correspond to points of $\mathbf{A}$;
- consider $<$ such that every branch of S is monotone;
- (3) $\Rightarrow \exists f : \mathbf{A}_< \to \mathbf{D}_{p+1}$;
- two endpoints of a branch of S have distinct images by f;
- $\Rightarrow \chi(H) \leq |\mathbf{D}_{p+1}|$.

$\lightning$

Proof

- (1) $\Rightarrow$ (2) already proved; (2) $\Rightarrow$ (3) is straightforward.
- (3) $\Rightarrow$ (1) by contradiction.

$\neg(1) \Rightarrow \mathrm{Incid}(\mathcal{C})$ does not have bounded expansion
$\Rightarrow \exists p : \chi(\mathrm{Incid}(\mathcal{C}) \,\widetilde{\nabla}\, p) = \infty.$

- $\exists H, S, \mathbf{A}$: $\chi(H) > |\mathbf{D}_{p+1}|$, $\delta(H) >$ maximum arity, S is a $\leq p$-subdivision of H and $S \subseteq \mathrm{Incid}(\mathbf{A}) \in \mathrm{Incid}(\mathcal{C})$;
- the branching vertices of S correspond to points of $\mathbf{A}$;
- consider $<$ such that every branch of S is monotone;
- (3) $\Rightarrow \exists f : \mathbf{A}_< \to \mathbf{D}_{p+1}$;
- two endpoints of a branch of S have distinct images by f;
- $\Rightarrow \chi(H) \leq |\mathbf{D}_{p+1}|$.

$\lightning$

Proof

- (1) $\Rightarrow$ (2) already proved; (2) $\Rightarrow$ (3) is straightforward.
- (3) $\Rightarrow$ (1) by contradiction.

$\neg(1) \Rightarrow \mathrm{Incid}(\mathcal{C})$ does not have bounded expansion

$\Rightarrow \exists p : \chi(\mathrm{Incid}(\mathcal{C}) \,\widetilde{\nabla}\, p) = \infty.$

- $\exists H, S, \mathbf{A}$: $\chi(H) > |\mathbf{D}_{p+1}|$, $\delta(H) >$ maximum arity, S is a $\leq p$-subdivision of H and $S \subseteq \mathrm{Incid}(\mathbf{A}) \in \mathrm{Incid}(\mathcal{C})$;
- the branching vertices of S correspond to points of $\mathbf{A}$;
- consider $<$ such that every branch of S is monotone;
- (3) $\Rightarrow \exists f : \mathbf{A}_< \to \mathbf{D}_{p+1}$;
- two endpoints of a branch of S have distinct images by f;
- $\Rightarrow \chi(H) \leq |\mathbf{D}_{p+1}|$.

$\lightning$

Proof

- (1) $\Rightarrow$ (2) already proved; (2) $\Rightarrow$ (3) is straightforward.
- (3) $\Rightarrow$ (1) by contradiction.

$\neg(1) \Rightarrow \mathrm{Incid}(\mathcal{C})$ does not have bounded expansion
$\Rightarrow \exists p : \chi(\mathrm{Incid}(\mathcal{C}) \,\widetilde{\nabla}\, p) = \infty.$

- $\exists H, S, \mathbf{A}$: $\chi(H) > |\mathbf{D}_{p+1}|$, $\delta(H) >$ maximum arity, S is a $\leq p$-subdivision of H and $S \subseteq \mathrm{Incid}(\mathbf{A}) \in \mathrm{Incid}(\mathcal{C})$;
- the branching vertices of S correspond to points of $\mathbf{A}$;
- consider $<$ such that every branch of S is monotone;
- (3) $\Rightarrow \exists f : \mathbf{A}_< \to \mathbf{D}_{p+1}$;
- two endpoints of a branch of S have distinct images by f;
- $\Rightarrow \chi(H) \leq |\mathbf{D}_{p+1}|$.

↯

Proof

- (1) $\Rightarrow$ (2) already proved; (2) $\Rightarrow$ (3) is straightforward.
- (3) $\Rightarrow$ (1) by contradiction.

$\neg(1) \Rightarrow$ $\mathrm{Incid}(\mathcal{C})$ does not have bounded expansion
$\Rightarrow$ $\exists p : \chi(\mathrm{Incid}(\mathcal{C}) \mathbin{\widetilde{\nabla}} p) = \infty$.

- $\exists H, S, \mathbf{A}$: $\chi(H) > |\mathbf{D}_{p+1}|$, $\delta(H) >$ maximum arity, S is a $\leq p$-subdivision of H and $S \subseteq \mathrm{Incid}(\mathbf{A}) \in \mathrm{Incid}(\mathcal{C})$;
- the branching vertices of S correspond to points of $\mathbf{A}$;
- consider $<$ such that every branch of S is monotone;
- (3) $\Rightarrow \exists f : \mathbf{A}_< \to \mathbf{D}_{p+1}$;
- two endpoints of a branch of S have distinct images by f;
- $\Rightarrow \chi(H) \leq |\mathbf{D}_{p+1}|$.

$\lightning$

Classes of graphs closed by subdivisions

27 Homomorphisms and CSP

28 Finite dualities

29 Restricted dualities

30 Bounded expansion classes

31 Classes of structures

32 Classes of graphs closed by subdivisions

Classes of graphs closed by subdivisions

Theorem

Let $\mathcal{C}$ be a class of undirected graphs closed by subdivisions. The following are equivalent:

1. *the class $\mathcal{C}$ has bounded expansion;*
2. *the class $\mathcal{C}$ has all restricted dualities;*
3. *for every odd integer g there exists a non-bipartite graph H_g with odd-girth at least g such that*

$$\forall G \in \mathcal{C}, \qquad \mathrm{og}(G) \geq g \implies G \to H_g$$

First order definable H-colorings

First order definable H-colorings

H-coloring is *first-order definable* on $\mathcal{C}$ if there exists a first-order formula Φ such that

$$\forall G \in \mathcal{C} \qquad (G \models \Phi) \iff (G \to H).$$

FOG Conjecture

Let $\mathcal{C}$ be a hereditary addable class of graphs closed by subdivisions. The following are equivalent:

- there exists in $\mathcal{C}$ first-order definable H-colorings for non bipartite H of arbitrarily large odd-girth;
- the class $\mathcal{C}$ has bounded expansion.

Ways to the conjecture

1. Erdős-Hajnal conjecture $\Longrightarrow$ FOG-conjecture;
2. Thomassen conjecture $\Longrightarrow$ FOG-conjecture.

Proof.

Only one direction. Two steps:

1. $\mathcal{C}$ is not somewhere dense:
   If $\mathcal{C} \mathbin{\widetilde{\nabla}} p = \mathcal{G}\mathit{raph}$, H is non-bipartite and $\mathrm{og}(H) > 2p + 1$, then H-coloring is not first-order definable in $\mathcal{C}$.
2. $\mathcal{C}$ has bounded expansion.

$\square$

Proof of first step

Assume $\exists \Phi : \forall G \in \mathcal{C}, \ (G \models \Phi) \iff (G \to H)$.

- $\neg\Phi$ preserved by homomorphisms on $\mathrm{Sub}_{2p}(\mathcal{G}raph)$
- homomorphism preservation theorem $\Rightarrow$

$$\exists \mathcal{F}, \qquad \forall F \in \mathcal{F} \ F \nrightarrow G^{(2p)} \iff G^{(2p)} \to H.$$

- graphs in $\mathcal{F}$ are non-bipartite;
- choose G, $\chi(G) > |H|$ and $\mathrm{og}(G) > \max_{F \in \mathcal{F}} \mathrm{og}(F)$.
- $\forall F \in \mathcal{F}, \ F \nrightarrow G^{(2p)} \Rightarrow G^{(2p)} \to H$.
- two branching vertices of $G^{(2p)}$ cannot be mapped to a same vertex

$\Rightarrow \chi(G) \leq |H|$ $\quad ↯$

Proof of first step

Assume $\exists\Phi : \ \forall G \in \mathcal{C}, \ (G \models \Phi) \iff (G \to H)$.

- $\neg\Phi$ preserved by homomorphisms on $\mathrm{Sub}_{2p}(\mathcal{G}raph)$
- homomorphism preservation theorem $\Rightarrow$

$$\exists \mathcal{F}, \qquad \forall F \in \mathcal{F} \ F \nrightarrow G^{(2p)} \iff G^{(2p)} \to H.$$

- graphs in $\mathcal{F}$ are non-bipartite;
- choose G, $\chi(G) > |H|$ and $\mathrm{og}(G) > \max_{F \in \mathcal{F}} \mathrm{og}(F)$.
- $\forall F \in \mathcal{F}, \ F \nrightarrow G^{(2p)} \Rightarrow G^{(2p)} \to H$.
- two branching vertices of $G^{(2p)}$ cannot be mapped to a same vertex

$\Rightarrow \chi(G) \leq |H|$ $\lightning$

Proof of first step

Assume $\exists\Phi : \ \forall G \in \mathcal{C}, \ (G \vDash \Phi) \iff (G \to H)$.

- $\neg\Phi$ preserved by homomorphisms on $\mathrm{Sub}_{2p}(\mathcal{G}raph)$
- homomorphism preservation theorem $\Rightarrow$

$$\exists\mathcal{F}, \qquad \forall F \in \mathcal{F}\ F \nrightarrow G^{(2p)} \iff G^{(2p)} \to H.$$

- graphs in $\mathcal{F}$ are non-bipartite;
- choose G, $\chi(G) > |H|$ and $\mathrm{og}(G) > \max_{F\in\mathcal{F}} \mathrm{og}(F)$.
- $\forall F \in \mathcal{F}, \ F \nrightarrow G^{(2p)} \Rightarrow G^{(2p)} \to H$.
- two branching vertices of $G^{(2p)}$ cannot be mapped to a same vertex

$\Rightarrow \chi(G) \leq |H|$ ↯

Proof of first step

Assume $\exists \Phi : \ \forall G \in \mathcal{C}, \ (G \vDash \Phi) \iff (G \to H)$.

- $\neg\Phi$ preserved by homomorphisms on $\mathrm{Sub}_{2p}(\mathcal{G}raph)$
- homomorphism preservation theorem $\Rightarrow$

$$\exists \mathcal{F}, \qquad \forall F \in \mathcal{F} \ F \nrightarrow G^{(2p)} \iff G^{(2p)} \to H.$$

- graphs in $\mathcal{F}$ are non-bipartite;
- choose G, $\chi(G) > |H|$ and $\mathrm{og}(G) > \max_{F \in \mathcal{F}} \mathrm{og}(F)$.
- $\forall F \in \mathcal{F}, \ F \nrightarrow G^{(2p)} \Rightarrow G^{(2p)} \to H$.
- two branching vertices of $G^{(2p)}$ cannot be mapped to a same vertex

$\Rightarrow \chi(G) \leq |H|$ $\quad ↯$

Proof of first step

Assume $\exists \Phi : \ \forall G \in \mathcal{C}, \ (G \models \Phi) \iff (G \to H)$.

- $\neg\Phi$ preserved by homomorphisms on $\mathrm{Sub}_{2p}(\mathcal{G}raph)$
- homomorphism preservation theorem $\Rightarrow$

$$\exists \mathcal{F}, \qquad \forall F \in \mathcal{F} \ F \nrightarrow G^{(2p)} \iff G^{(2p)} \to H.$$

- graphs in $\mathcal{F}$ are non-bipartite;
- choose G, $\chi(G) > |H|$ and $\mathrm{og}(G) > \max_{F \in \mathcal{F}} \mathrm{og}(F)$.
- $\forall F \in \mathcal{F}, \ F \nrightarrow G^{(2p)} \Rightarrow G^{(2p)} \to H$.
- two branching vertices of $G^{(2p)}$ cannot be mapped to a same vertex

$\Rightarrow \chi(G) \leq |H| \qquad \lightning$

Proof of first step

Assume $\exists \Phi : \ \forall G \in \mathcal{C}, \ (G \models \Phi) \iff (G \to H)$.

- $\neg\Phi$ preserved by homomorphisms on $\mathrm{Sub}_{2p}(\mathcal{G}raph)$
- homomorphism preservation theorem $\Rightarrow$

$$\exists \mathcal{F}, \qquad \forall F \in \mathcal{F} \ F \nrightarrow G^{(2p)} \iff G^{(2p)} \to H.$$

- graphs in $\mathcal{F}$ are non-bipartite;
- choose G, $\chi(G) > |H|$ and $\mathrm{og}(G) > \max_{F \in \mathcal{F}} \mathrm{og}(F)$.
- $\forall F \in \mathcal{F}, \ F \nrightarrow G^{(2p)} \Rightarrow G^{(2p)} \to H$.
- two branching vertices of $G^{(2p)}$ cannot be mapped to a same vertex

$\Rightarrow \chi(G) \leq |H| \qquad \lightning$

Proof of first step

Assume $\exists\Phi: \ \forall G \in \mathcal{C}, \ (G \models \Phi) \iff (G \to H)$.

- $\neg\Phi$ preserved by homomorphisms on $\mathrm{Sub}_{2p}(\mathcal{G}raph)$
- homomorphism preservation theorem $\Rightarrow$

$$\exists\mathcal{F}, \qquad \forall F \in \mathcal{F} \ F \nrightarrow G^{(2p)} \iff G^{(2p)} \to H.$$

- graphs in $\mathcal{F}$ are non-bipartite;
- choose G, $\chi(G) > |H|$ and $\mathrm{og}(G) > \max_{F\in\mathcal{F}} \mathrm{og}(F)$.
- $\forall F \in \mathcal{F}, \ F \nrightarrow G^{(2p)} \Rightarrow G^{(2p)} \to H$.
- two branching vertices of $G^{(2p)}$ cannot be mapped to a same vertex

$\Rightarrow \ \chi(G) \leq |H| \qquad ↯$

Proof of second step

- If $\mathcal{C}$ does not have bounded expansion, $\exists p : \chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$.
- $\exists H, \Phi$: $\mathrm{og}(H) \geq 2p+5$ and $G \models \Phi \iff G \nrightarrow H$;
- By HPT, $\exists \mathcal{F} \subseteq \mathcal{C}$ finite s.t.
  $\forall G \in \mathcal{C}, \qquad (\exists F \in \mathcal{F} : F \to G) \iff G \nrightarrow H$;
- $\mathcal{F}$ includes no bipartite graph;
- Let $g = \max_{F \in \mathcal{F}} \mathrm{og}(F)$. As $\chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$,
  - E-H conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g+2, \chi(G) > |H|$;
  - T conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g+2, \delta(G) > f(|H|)$
    $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p + \frac{1}{2}, \mathrm{og}(G) \geq g+2, \chi(G) > |H|$;
- $\forall F \in \mathcal{F}, F \nrightarrow G^{2p+2}$ hence $G^{(2p+2)} \to H$;
- two branching vertices of $G^{(2p+2)}$ cannot have same image;
  $\Rightarrow \chi(G) \leq |H| \qquad ↯$

Proof of second step

- If $\mathcal{C}$ does not have bounded expansion, $\exists p : \chi(\mathcal{C} \mathbin{\widetilde{\nabla}} p) = \infty$.
- $\exists H, \Phi$: $\mathrm{og}(H) \geq 2p + 5$ and $G \models \Phi \iff G \nrightarrow H$;
- By HPT, $\exists \mathcal{F} \subseteq \mathcal{C}$ finite s.t.
  $\forall G \in \mathcal{C}, \quad (\exists F \in \mathcal{F} : F \to G) \iff G \nrightarrow H$;
- $\mathcal{F}$ includes no bipartite graph;
- Let $g = \max_{F \in \mathcal{F}} \mathrm{og}(F)$. As $\chi(\mathcal{C} \mathbin{\widetilde{\nabla}} p) = \infty$,
  - E-H conj. $\Rightarrow \exists G \in \mathcal{C} \mathbin{\widetilde{\nabla}} p, \mathrm{og}(G) \geq g + 2, \chi(G) > |H|$;
  - T conj. $\Rightarrow \exists G \in \mathcal{C} \mathbin{\widetilde{\nabla}} p, \mathrm{og}(G) \geq g + 2, \delta(G) > f(|H|)$
    $\Rightarrow \exists G \in \mathcal{C} \mathbin{\widetilde{\nabla}} p + \frac{1}{2}, \mathrm{og}(G) \geq g + 2, \chi(G) > |H|$;
- $\forall F \in \mathcal{F}, F \nrightarrow G^{2p+2}$ hence $G^{(2p+2)} \to H$;
- two branching vertices of $G^{(2p+2)}$ cannot have same image;
  $\Rightarrow \chi(G) \leq |H| \qquad ↯$

Proof of second step

- If $\mathcal{C}$ does not have bounded expansion, $\exists p : \chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$.
- $\exists H, \Phi$: $\mathrm{og}(H) \geq 2p + 5$ and $G \models \Phi \iff G \nrightarrow H$;
- By HPT, $\exists \mathcal{F} \subseteq \mathcal{C}$ finite s.t.
  $\forall G \in \mathcal{C}, \qquad (\exists F \in \mathcal{F} : F \to G) \iff G \nrightarrow H$;
- $\mathcal{F}$ includes no bipartite graph;
- Let $g = \max_{F \in \mathcal{F}} \mathrm{og}(F)$. As $\chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$,
  - E-H conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g + 2, \chi(G) > |H|$;
  - T conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g + 2, \delta(G) > f(|H|)$
    $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p + \frac{1}{2}, \mathrm{og}(G) \geq g + 2, \chi(G) > |H|$;
- $\forall F \in \mathcal{F}, F \nrightarrow G^{2p+2}$ hence $G^{(2p+2)} \to H$;
- two branching vertices of $G^{(2p+2)}$ cannot have same image;
  $\Rightarrow \chi(G) \leq |H|$ $\lightning$

Proof of second step

- If $\mathcal{C}$ does not have bounded expansion, $\exists p : \chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$.
- $\exists H, \Phi$: $\mathrm{og}(H) \geq 2p+5$ and $G \models \Phi \iff G \nrightarrow H$;
- By HPT, $\exists \mathcal{F} \subseteq \mathcal{C}$ finite s.t.
  $\forall G \in \mathcal{C}, \qquad (\exists F \in \mathcal{F} : F \to G) \iff G \nrightarrow H$;
- $\mathcal{F}$ includes no bipartite graph;
- Let $g = \max_{F \in \mathcal{F}} \mathrm{og}(F)$. As $\chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$,
  - E-H conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g+2, \chi(G) > |H|$;
  - T conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g+2, \delta(G) > f(|H|)$
    $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p + \frac{1}{2}, \mathrm{og}(G) \geq g+2, \chi(G) > |H|$;
- $\forall F \in \mathcal{F}, F \nrightarrow G^{2p+2}$ hence $G^{(2p+2)} \to H$;
- two branching vertices of $G^{(2p+2)}$ cannot have same image;
  $\Rightarrow \chi(G) \leq |H| \qquad ↯$

Proof of second step

- If $\mathcal{C}$ does not have bounded expansion, $\exists p : \chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$.
- $\exists H, \Phi$: $\mathrm{og}(H) \geq 2p + 5$ and $G \models \Phi \iff G \nrightarrow H$;
- By HPT, $\exists \mathcal{F} \subseteq \mathcal{C}$ finite s.t.
  $\forall G \in \mathcal{C}, \qquad (\exists F \in \mathcal{F} : F \to G) \iff G \nrightarrow H$;
- $\mathcal{F}$ includes no bipartite graph;
- Let $g = \max_{F \in \mathcal{F}} \mathrm{og}(F)$. As $\chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$,
  - E-H conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g + 2, \chi(G) > |H|$;
  - T conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g + 2, \delta(G) > f(|H|)$
    $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p + \frac{1}{2}, \mathrm{og}(G) \geq g + 2, \chi(G) > |H|$;
- $\forall F \in \mathcal{F}, F \nrightarrow G^{2p+2}$ hence $G^{(2p+2)} \to H$;
- two branching vertices of $G^{(2p+2)}$ cannot have same image;
  $\Rightarrow \chi(G) \leq |H|$ ↯

Proof of second step

- If $\mathcal{C}$ does not have bounded expansion, $\exists p : \chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$.
- $\exists H, \Phi$: $\mathrm{og}(H) \geq 2p + 5$ and $G \models \Phi \iff G \nrightarrow H$;
- By HPT, $\exists \mathcal{F} \subseteq \mathcal{C}$ finite s.t.
  $\forall G \in \mathcal{C}, \qquad (\exists F \in \mathcal{F} : F \rightarrow G) \iff G \nrightarrow H$;
- $\mathcal{F}$ includes no bipartite graph;
- Let $g = \max_{F \in \mathcal{F}} \mathrm{og}(F)$. As $\chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$,
  - E-H conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g + 2, \chi(G) > |H|$;
  - T conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g + 2, \delta(G) > f(|H|)$
    $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p + \frac{1}{2}, \mathrm{og}(G) \geq g + 2, \chi(G) > |H|$;
- $\forall F \in \mathcal{F}, F \nrightarrow G^{2p+2}$ hence $G^{(2p+2)} \rightarrow H$;
- two branching vertices of $G^{(2p+2)}$ cannot have same image;
  $\Rightarrow \chi(G) \leq |H| \qquad \lightning$

Proof of second step

- If $\mathcal{C}$ does not have bounded expansion, $\exists p : \chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$.
- $\exists H, \Phi$: $\mathrm{og}(H) \geq 2p + 5$ and $G \models \Phi \iff G \nrightarrow H$;
- By HPT, $\exists \mathcal{F} \subseteq \mathcal{C}$ finite s.t.
  $\forall G \in \mathcal{C}, \qquad (\exists F \in \mathcal{F} : F \to G) \iff G \nrightarrow H$;
- $\mathcal{F}$ includes no bipartite graph;
- Let $g = \max_{F \in \mathcal{F}} \mathrm{og}(F)$. As $\chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$,
  - E-H conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g + 2, \chi(G) > |H|$;
  - T conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g + 2, \delta(G) > f(|H|)$
    $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p + \frac{1}{2}, \mathrm{og}(G) \geq g + 2, \chi(G) > |H|$;
- $\forall F \in \mathcal{F}, F \nrightarrow G^{2p+2}$ hence $G^{(2p+2)} \to H$;
- two branching vertices of $G^{(2p+2)}$ cannot have same image;
  $\Rightarrow \chi(G) \leq |H| \qquad \lightning$

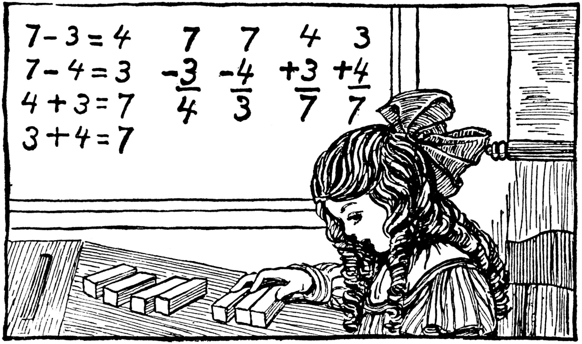
Proof of second step

- If $\mathcal{C}$ does not have bounded expansion, $\exists p : \chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$.
- $\exists H, \Phi$: $\mathrm{og}(H) \geq 2p + 5$ and $G \models \Phi \iff G \nrightarrow H$;
- By HPT, $\exists \mathcal{F} \subseteq \mathcal{C}$ finite s.t.
  $\forall G \in \mathcal{C}, \qquad (\exists F \in \mathcal{F} : F \to G) \iff G \nrightarrow H$;
- $\mathcal{F}$ includes no bipartite graph;
- Let $g = \max_{F \in \mathcal{F}} \mathrm{og}(F)$. As $\chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$,
  - E-H conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g + 2, \chi(G) > |H|$;
  - T conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g + 2, \delta(G) > f(|H|)$
    $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p + \frac{1}{2}, \mathrm{og}(G) \geq g + 2, \chi(G) > |H|$;
- $\forall F \in \mathcal{F}, F \nrightarrow G^{2p+2}$ hence $G^{(2p+2)} \to H$;
- two branching vertices of $G^{(2p+2)}$ cannot have same image;
  $\Rightarrow \chi(G) \leq |H| \qquad \lightning$

Proof of second step

- If $\mathcal{C}$ does not have bounded expansion, $\exists p : \chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$.
- $\exists H, \Phi$: $\mathrm{og}(H) \geq 2p + 5$ and $G \models \Phi \iff G \nrightarrow H$;
- By HPT, $\exists \mathcal{F} \subseteq \mathcal{C}$ finite s.t. $\forall G \in \mathcal{C}, \qquad (\exists F \in \mathcal{F} : F \to G) \iff G \nrightarrow H$;
- $\mathcal{F}$ includes no bipartite graph;
- Let $g = \max_{F \in \mathcal{F}} \mathrm{og}(F)$. As $\chi(\mathcal{C} \,\widetilde{\nabla}\, p) = \infty$,
  - E-H conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g + 2, \chi(G) > |H|$;
  - T conj. $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p, \mathrm{og}(G) \geq g + 2, \delta(G) > f(|H|)$ $\Rightarrow \exists G \in \mathcal{C} \,\widetilde{\nabla}\, p + \frac{1}{2}, \mathrm{og}(G) \geq g + 2, \chi(G) > |H|$;
- $\forall F \in \mathcal{F}, F \nrightarrow G^{2p+2}$ hence $G^{(2p+2)} \to H$;
- two branching vertices of $G^{(2p+2)}$ cannot have same image; $\Rightarrow \chi(G) \leq |H| \qquad \lightning$

Counting



Motivation

33 Motivation

34 Sunflowers

35 Main result

36 Sketch of the proof

Counting Statistics

$$\begin{tikzcd} F_1 \arrow[dr] & \\ F_2 \arrow[r] & G \\ F_3 \arrow[ur] & \end{tikzcd}$$

- Dense graphs:

$$t(F, G) = \frac{\hom(F, G)}{|G|^{|F|}}$$

$$\begin{tikzcd} F_1 \arrow[dr, hook] & \\ F_2 \arrow[r, hook] & G \\ F_3 \arrow[ur, hook] & \end{tikzcd}$$

- Very sparse graphs:

$$\mathrm{dens}(F, G) = \frac{(\# F \subseteq G)}{|G|}$$

Counting Statistics

$$\begin{tikzcd} F_1 \arrow[dr] & \\ F_2 \arrow[r] & G \\ F_3 \arrow[ur] & \end{tikzcd}$$

- Dense graphs:

$$t(F, G) = \frac{\hom(F, G)}{|G|^{|F|}}$$

$$\begin{tikzcd} F_1 \arrow[dr, hook] & \\ F_2 \arrow[r, hook] & G \\ F_3 \arrow[ur, hook] & \end{tikzcd}$$

- Very sparse graphs:

$$\mathrm{dens}(F, G) = \frac{(\# F \subseteq G)}{|G|}$$

Counting Statistics

$$\begin{tikzcd} F_1 \arrow[dr] & \\ F_2 \arrow[r] & G \\ F_3 \arrow[ur] & \end{tikzcd}$$

- Dense graphs:

$$t(F, G) = \frac{\hom(F, G)}{|G|^{|F|}}$$

$$\begin{tikzcd} F_1 \arrow[dr, hook] & \\ F_2 \arrow[r, hook] & G \\ F_3 \arrow[ur, hook] & \end{tikzcd}$$

- Very sparse graphs:

$$\mathrm{dens}(F, G) = \frac{(\#F \subseteq G)}{|G|^{1}}$$

Main Result

Looking for the exponent / the degree of freedom

- How is $\frac{\log(\#F \subseteq G)}{\log|G|}$ bounded when G is restricted to a class $\mathcal{C}$?
- How much vertices can be chosen independently when looking for a copy of F?

Main Result

Looking for the exponent / the degree of freedom

- How is $\frac{\log(\#F \subseteq G)}{\log|G|}$ bounded when G is restricted to a class $\mathcal{C}$?
- How much vertices can be chosen independently when looking for a copy of F?

Theorem

For every infinite class of graphs $\mathcal{C}$ and every graph F

$$\lim_{i\to\infty} \limsup_{G\in\mathcal{C}\,\triangledown\, i} \frac{\log(\#F \subseteq G)}{\log|G|} \in \{-\infty, 0, 1, \ldots, \alpha(F), |F|\},$$

where $\alpha(F)$ is the stability number of F.

Sunflowers

33 Motivation

34 Sunflowers

35 Main result

36 Sketch of the proof

Generalized Sunflowers

Question

Assume G contains a large number of copies of F.
Then G contains many copies that form a regular structure?

Generalized Sunflowers

Question

Assume G contains a large number of copies of F.
Then G contains many copies that form a regular structure?

(k, F)-sunflower $(C, \mathcal{F}_1, \ldots, \mathcal{F}_k)$

Let F, G be graphs. A *(k, F)-sunflower* in G is a $(k+1)$-tuple $(C, \mathcal{F}_1, \ldots, \mathcal{F}_k)$, such that $C \subseteq V(G), \mathcal{F}_i \subseteq \mathcal{P}(V(G))$, the sets in $\{C\} \cup \bigcup_i \mathcal{F}_i$ are pairwise disjoints and there exists a partition $(K, Y_1, \ldots, Y_k)$ of $V(F)$ so that

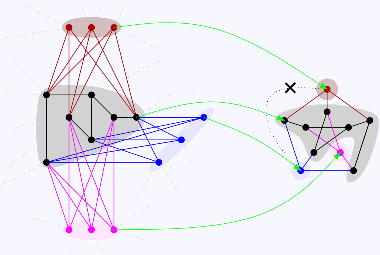
- $\forall i \neq j,\ \omega(Y_i, Y_j) = \emptyset$,
- $G[C] \approx F[K]$,
- $\forall X_i \in \mathcal{F}_i, G[X_i] \approx F[Y_i]$,
- $\forall (X_1, \ldots, X_k) \in \mathcal{F}_1 \times \cdots \times \mathcal{F}_k, G[C \cup X_1 \cup \cdots \cup X_k] \approx F$.

Generalized Sunflowers

Question

Assume G contains a large number of copies of F.
Then G contains many copies that form a regular structure?

(k, F)-sunflower $(C, \mathcal{F}_1, \ldots, \mathcal{F}_k)$



$$\forall X_1 \in \mathcal{F}_1, \ldots \forall X_k \in \mathcal{F}_k$$

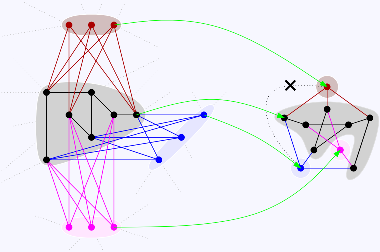
$$G[C \cup X_1 \cup \cdots \cup X_k] \approx F$$

Generalized Sunflowers

Question

Assume G contains a large number of copies of F.
Then G contains many copies that form a regular structure?

(k, F)-sunflower $(C, \mathcal{F}_1, \ldots, \mathcal{F}_k)$



$$\forall X_1 \in \mathcal{F}_1, \ldots \forall X_k \in \mathcal{F}_k$$

$$G[C \cup X_1 \cup \cdots \cup X_k] \approx F$$

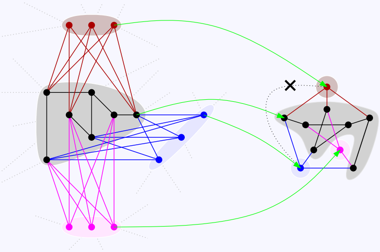
$$\Rightarrow k \leq \alpha(F)$$

Generalized Sunflowers

Question

Assume G contains a large number of copies of F.
Then G contains many copies that form a regular structure?

(k, F)-sunflower $(C, \mathcal{F}_1, \ldots, \mathcal{F}_k)$



$$\forall X_1 \in \mathcal{F}_1, \ldots \forall X_k \in \mathcal{F}_k$$

$$G[C \cup X_1 \cup \cdots \cup X_k] \approx F$$

$$\Rightarrow k \leq \alpha(F) \text{ and}$$

$$(\# F \subseteq G) \geq \prod_{i=1}^{k} |\mathcal{F}_i|.$$

Main result

33 Motivation

34 Sunflowers

35 Main result

36 Sketch of the proof

Clearing & Stepping Up

Let F be a graph of order p, let $k \in \mathbb{N}$ and let $0 < \epsilon < 1$.
For every graph G such that $(\#F \subseteq G) > |G|^{k+\epsilon}$ there exists in G a $(k+1, F)$-sunflower$(C, \mathcal{F}_1, \ldots, \mathcal{F}_{k+1})$ with

$$\min_i |\mathcal{F}_i| \geq \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\epsilon}} \right)^{\tau(\epsilon,p)}$$

Clearing & Stepping Up

Let F be a graph of order p, let $k \in \mathbb{N}$ and let $0 < \epsilon < 1$.
For every graph G such that $(\#F \subseteq G) > |G|^{k+\epsilon}$ there exists in G a $(k+1, F)$-sunflower$(C, \mathcal{F}_1, \ldots, \mathcal{F}_{k+1})$ with

$$\min_i |\mathcal{F}_i| \geq \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\epsilon}} \right)^{\tau(\epsilon,p)}$$

Clearing & Stepping Up

Let F be a graph of order p, let $k \in \mathbb{N}$ and let $0 < \epsilon < 1$.
For every graph G such that $(\#F \subseteq G) > |G|^{k+\epsilon}$ there exists in G a $(k+1, F)$-sunflower$(C, \mathcal{F}_1, \ldots, \mathcal{F}_{k+1})$ with

$$\min_i |\mathcal{F}_i| \geq \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\epsilon}} \right)^{\tau(\epsilon,p)}$$

Clearing & Stepping Up

Let F be a graph of order p, let $k \in \mathbb{N}$ and let $0 < \epsilon < 1$.
For every graph G such that $(\#F \subseteq G) > |G|^{k+\epsilon}$ there exists in G a $(k+1, F)$-sunflower$(C, \mathcal{F}_1, \ldots, \mathcal{F}_{k+1})$ with

$$\min_i |\mathcal{F}_i| \geq \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\epsilon}} \right)^{\tau(\epsilon,p)}$$

Hence $\exists G' \subseteq G$ such that

$$|G'| \geq (k+1) \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\epsilon}} \right)^{\tau(\epsilon,p)}$$

and $$(\#F \subseteq G') \geq \left(\frac{|G'| - |F|}{k+1} \right)^{k+1}.$$

Clearing & Stepping Up

Let F be a graph of order p, let $k \in \mathbb{N}$ and let $0 < \epsilon < 1$.
For every graph G such that $(\#F \subseteq G) > |G|^{k+\epsilon}$ there exists in G a $(k+1, F)$-sunflower$(C, \mathcal{F}_1, \dots, \mathcal{F}_{k+1})$ with

$$\min_i |\mathcal{F}_i| \geq \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\epsilon}} \right)^{\tau(\epsilon,p)}$$

Hence $\exists G' \subseteq G$ such that

$$|G'| \geq (k+1) \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\epsilon}} \right)^{\tau(\epsilon,p)}$$

and $$(\#F \subseteq G') \geq \left(\frac{|G'| - |F|}{k+1} \right)^{k+1} .$$

Clearing & Stepping Up

Let F be a graph of order p, let $k \in \mathbb{N}$ and let $0 < \epsilon < 1$.
For every graph G such that $(\#F \subseteq G) > |G|^{k+\epsilon}$ there exists in G a $(k+1, F)$-sunflower$(C, \mathcal{F}_1, \ldots, \mathcal{F}_{k+1})$ with

$$\min_i |\mathcal{F}_i| \geq \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\epsilon}} \right)^{\tau(\epsilon,p)}$$

Hence $\exists G' \subseteq G$ such that

$$|G'| \geq (k+1) \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\epsilon}} \right)^{\tau(\epsilon,p)}$$

and $$(\#F \subseteq G') \geq \left(\frac{|G'| - |F|}{k+1} \right)^{k+1}.$$

Clearing & Stepping Up

Let F be a graph of order p, let $k \in \mathbb{N}$ and let $0 < \epsilon < 1$.
For every graph G such that $(\#F \subseteq G) > |G|^{k+\epsilon}$ there exists in G a $(k+1, F)$-sunflower$(C, \mathcal{F}_1, \ldots, \mathcal{F}_{k+1})$ with

$$\min_i |\mathcal{F}_i| \geq \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\epsilon}} \right)^{\tau(\epsilon,p)}$$

Hence $\exists G' \subseteq G$ such that

$$|G'| \geq (k+1) \left(\frac{|G|}{\binom{\chi_p(G)}{p}^{1/\epsilon}} \right)^{\tau(\epsilon,p)}$$

and $$(\#F \subseteq G') \geq \left(\frac{|G'| - |F|}{k+1} \right)^{k+1}.$$

Consequence: Degrees of Freedom

Theorem

If $\mathcal{C}$ is an infinite hereditary class such that for every integer p

$$\limsup_{G \in \mathcal{C}} \frac{\log \chi_p(G)}{\log |G|} = 0$$

then for every graph F

$$\limsup_{G \in \mathcal{C}} \frac{\log(\# F \subseteq G)}{\log |G|} \in \{-\infty, 0, 1, \dots, \alpha(F)\}.$$

Consequence: Degrees of Freedom

Theorem

If $\mathcal{C}$ is an infinite nowhere dense hereditary class

then for every graph F

$$\limsup_{G\in\mathcal{C}} \frac{\log(\#F \subseteq G)}{\log|G|} \in \{-\infty, 0, 1, \ldots, \alpha(F)\}.$$

Consequence: Degrees of Freedom

Theorem

If $\mathcal{C}$ is an infinite nowhere dense hereditary class then for every graph F

$$\limsup_{G\in\mathcal{C}} \frac{\log(\#F \subseteq G)}{\log|G|} \in \{-\infty, 0, 1, \ldots, \alpha(F)\}.$$

Consequence: Degrees of Freedom

Theorem

If $\mathcal{C}$ is an infinite nowhere dense hereditary class then for every graph F

$$\limsup_{G \in \mathcal{C}} \frac{\log(\#F \subseteq G)}{\log |G|} \in \{-\infty, 0, 1, \dots, \alpha(F)\}.$$

Hence for every infinite nowhere dense class $\mathcal{C}$

$$\lim_{i \to \infty} \limsup_{G \in \mathcal{C} \,\triangledown\, i} \frac{\log(\#F \subseteq G)}{\log |G|} \in \{-\infty, 0, 1, \dots, \alpha(F)\}.$$

Consequence: Degrees of Freedom

Theorem

If $\mathcal{C}$ is an infinite nowhere dense hereditary class then for every graph F

$$\limsup_{G \in \mathcal{C}} \frac{\log(\#F \subseteq G)}{\log |G|} \in \{-\infty, 0, 1, \ldots, \alpha(F)\}.$$

Hence for every infinite nowhere dense class $\mathcal{C}$

$$\lim_{i \to \infty} \limsup_{G \in \mathcal{C} \,\nabla\, i} \frac{\log(\#F \subseteq G)}{\log |G|} \in \{-\infty, 0, 1, \ldots, \alpha(F)\}.$$

Although if $\mathcal{C}$ is an infinite somewhere dense class,

$$\lim_{i \to \infty} \limsup_{G \in \mathcal{C} \,\nabla\, i} \frac{\log(\#F \subseteq G)}{\log |G|} = |F|.$$

Sketch of the proof

33 Motivation

34 Sunflowers

35 Main result

36 Sketch of the proof

Sketch of the Proof

To be proved

Let F be a graph of order p, let $k \in \mathbb{N}$ and let $0 < \epsilon < 1$.
For every graph G such that $(\#F \subseteq G) > |G|^{k+\epsilon}$ there exists in G a $(k+1, F)$-sunflower$(C, \mathcal{F}_1, \dots, \mathcal{F}_{k+1})$ with

$$\min_i |\mathcal{F}_i| \geq \left(\frac{|G|}{c_1(p)\binom{\chi_p(G)}{p}} \right)^{c_2(p)\epsilon^p}$$

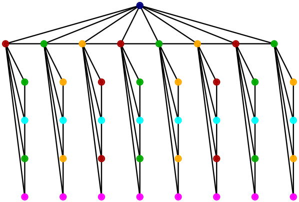
Steps

- Reduction: general graphs $\to$ graphs with bounded tree-depth,
- Reduction: graphs with bounded tree-depth $\to$ colored forests,
- Proof for colored forests.

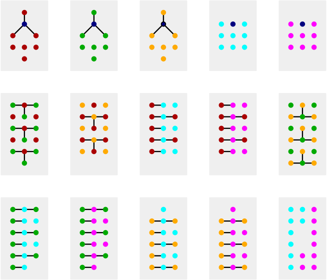
Reduction to bounded tree-depth

$\chi_p(G)$ colors

$\binom{\chi_p(G)}{p}$ subgraphs

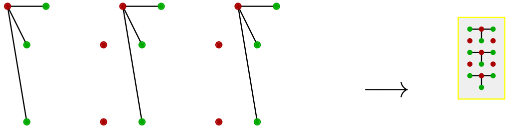


$\longrightarrow$



Reduction to bounded tree-depth

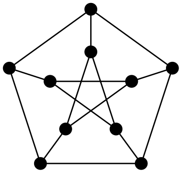
$\chi_p(G)$ colors $\qquad\qquad \binom{\chi_p(G)}{p}$ subgraphs



$\implies$ G has and induced subgraph G' such that $(\#F \subseteq G') \geq (\#F \subseteq G)/\binom{\chi_p(G)}{p}$ and $\mathrm{td}(G') \leq p$.

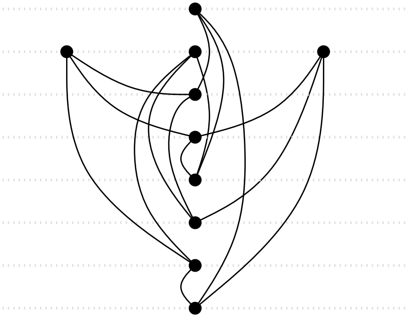
Reduction to colored forests

- Color coding



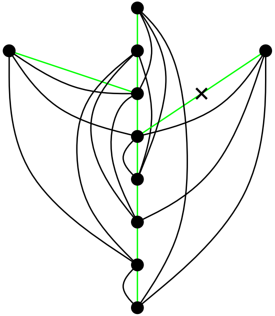
Reduction to colored forests

- Color coding $|F| = p = \#\text{levels} \quad \to \quad \leq p^p$ possibilities



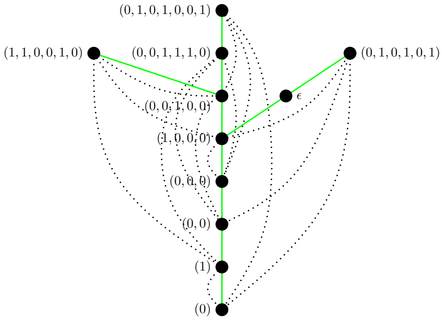
Reduction to colored forests

- Color coding $|F| = p = \#\text{levels} \quad \rightarrow \quad \leq c_1(p)$ possibilities



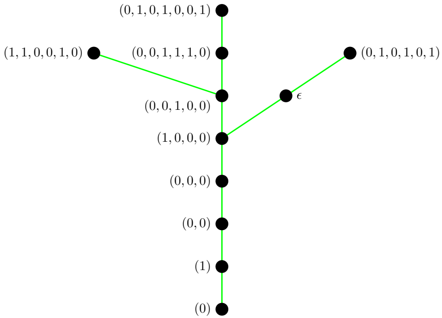
Reduction to colored forests

- Color coding $|F| = p = \#\text{levels} \quad \rightarrow \quad \leq c_1(p)$ possibilities



Reduction to colored forests

- Color coding $|F| = p = \#\text{levels} \quad \rightarrow \quad \leq c_1(p)$ possibilities



Sketch of the proof for colored forests

Proof by induction on the height of the forest F.

1. Determine where the components of F have to be mapped to get a positive fraction of the copies and some regularity,
2. Partition the components of F depending on the type of images,
3. Select a large "regular" subtree while non decreasing the logarithmic density of copies of F,
4. Use induction to find a $(k+1, F)$-sunflower

$\square$

Examples of classes with bounded expansion

Overview

37 Overview

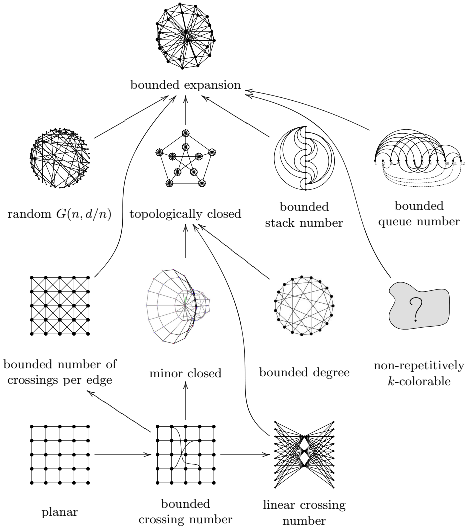
38 Random graphs

39 Queue number

40 Stack number

41 Non repetitive colorings

Overview


bounded expansion
random $G(n, d/n)$
topologically closed
bounded
stack number
bounded
queue number
bounded number of
crossings per edge
minor closed
bounded degree
non-repetitively
k-colorable
?
planar
bounded
crossing number
linear crossing
number

Random graphs

37 Overview

38 Random graphs

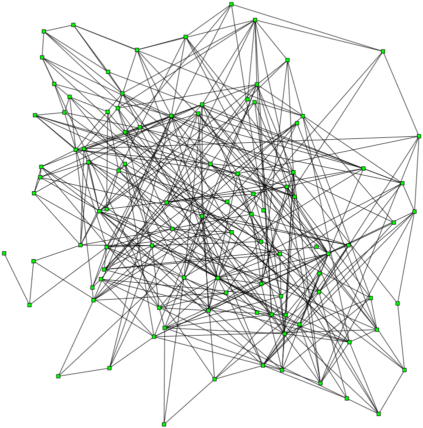
39 Queue number

40 Stack number

41 Non repetitive colorings

Random graphs (Erdős-Rényi model)

$G(n,p)$: graph with n vertices, each edge appears independently with probability $p = p(n)$.



First properties

- $\forall c > 1\ \exists \delta(c)$ such that

$$\delta(c)\sqrt{n} \leq \mathrm{h}(G(n, c/n)) \leq 2\sqrt{cn} \quad \text{(a.a.s.)}$$

- If $1 + \epsilon \leq (n-1)p = o(\sqrt{n})$ then

$$\mathrm{h}_t(G(n,p)) \approx \Delta(G(n,p)) = \Theta(\log n / \log\log n) \quad \text{(a.a.s.)}$$

- the expected number of cycles of length t in $G(n, c/n)$ is at most $(e^2c/2)^t \Rightarrow \mathrm{E}(\omega(G \,\widetilde{\nabla}\, d)) \approx (Ac)^{2d}$.

First properties

- $\forall c > 1\ \exists \delta(c)$ such that

$$\delta(c)\sqrt{n} \leq \mathrm{h}(G(n, c/n)) \leq 2\sqrt{cn} \quad \text{(a.a.s.)}$$

- If $1 + \epsilon \leq (n-1)p = o(\sqrt{n})$ then

$$\mathrm{h}_t(G(n,p)) \approx \Delta(G(n,p)) = \Theta(\log n / \log\log n) \quad \text{(a.a.s.)}$$

- the expected number of cycles of length t in $G(n, c/n)$ is at most $(e^2c/2)^t \ \Rightarrow\ \mathrm{E}(\omega(G \,\widetilde{\nabla}\, d)) \approx (Ac)^{2d}$.

First properties

- $\forall c > 1\ \exists \delta(c)$ such that
$$\delta(c)\sqrt{n} \leq \mathrm{h}(G(n, c/n)) \leq 2\sqrt{cn} \qquad \text{(a.a.s.)}$$
- If $1 + \epsilon \leq (n-1)p = o(\sqrt{n})$ then
$$\mathrm{h}_t(G(n,p)) \approx \Delta(G(n,p)) = \Theta(\log n / \log\log n) \qquad \text{(a.a.s.)}$$
- the expected number of cycles of length t in $G(n, c/n)$ is at most $(e^2c/2)^t \;\Rightarrow\; \mathrm{E}(\omega(G \,\widetilde{\nabla}\, d)) \approx (Ac)^{2d}$.

Another characterization of bounded expansion

Theorem

A class $\mathcal{C}$ of graphs has bounded expansion if, and only if, there exists functions $F_{\nabla}, F_{\mathrm{prop}}, F_{\mathrm{deg}} : \mathbb{R}^+ \to \mathbb{R}$ such that:

$$\forall r \in \mathbb{N},\ \forall H \subseteq G \in \mathcal{C}, \quad \widetilde{\nabla}_r(H) > F_{\nabla}(r) \Longrightarrow \frac{|H|}{|G|} > F_{\mathrm{prop}}(r)$$

$$\forall \epsilon > 0, \qquad \liminf_{G \in \mathcal{C}} \frac{|\{v \in G : \mathrm{d}(v) \geq F_{\mathrm{deg}}(\epsilon)\}|}{|G|} \leq \epsilon$$

Random graphs

Lemma

A.a.s. every subgraph H of $G(n, d/n)$ with $t \leq (4d)^{-(1+1/\epsilon)}n$ vertices satisfies $\widetilde{\nabla}_0(H) \leq 1 + \epsilon$.

Lemma

If $\widetilde{\nabla}_r(G) > 2$ then

$$\widetilde{\nabla}_0(G) > 1 + \frac{1}{2r+1} \ .$$

Corollary

Every subgraph H of $G(n, d/n)$ a.a.s. satisfies:

$$\widetilde{\nabla}_r(H) > 2 \Longrightarrow |H| > (4d)^{-(1+\frac{1}{2r+1})}|G| \ .$$

Random graphs

Lemma

A.a.s. every subgraph H of $G(n, d/n)$ with $t \leq (4d)^{-(1+1/\epsilon)}n$ vertices satisfies $\widetilde{\nabla}_0(H) \leq 1 + \epsilon$.

Lemma

If $\widetilde{\nabla}_r(G) > 2$ then

$$\widetilde{\nabla}_0(G) > 1 + \frac{1}{2r+1}\ .$$

Corollary

Every subgraph H of $G(n, d/n)$ a.a.s. satisfies:

$$\widetilde{\nabla}_r(H) > 2 \implies |H| > (4d)^{-(1+\frac{1}{2r+1})}|G|\ .$$

Random graphs

Lemma

A.a.s. every subgraph H of $G(n, d/n)$ with $t \leq (4d)^{-(1+1/\epsilon)}n$ vertices satisfies $\widetilde{\nabla}_0(H) \leq 1 + \epsilon$.

Lemma

If $\widetilde{\nabla}_r(G) > 2$ then

$$\widetilde{\nabla}_0(G) > 1 + \frac{1}{2r+1} \ .$$

Corollary

Every subgraph H of $G(n, d/n)$ a.a.s. satisfies:

$$\widetilde{\nabla}_r(H) > 2 \Longrightarrow |H| > (4d)^{-(1+\frac{1}{2r+1})}|G| \ .$$

Random graphs

Lemma

Let $\alpha > 1$ and let $c_\alpha = 4e\alpha^{-4\alpha d}$.
Asymptotically almost surely there are at most $c_\alpha n$ vertices of $G(n, d/n)$ with degree greater than $8\alpha d$.

Theorem

For each $d > 0$ there exists a class $\mathcal{R}_d$ with bounded expansion such that $G(n, d/n)$ a.a.s. belongs to $\mathcal{R}_d$.

Random graphs

Lemma

Let $\alpha > 1$ and let $c_\alpha = 4e\alpha^{-4\alpha d}$.
Asymptotically almost surely there are at most $c_\alpha n$ vertices of $G(n, d/n)$ with degree greater than $8\alpha d$.

Theorem

For each $d > 0$ there exists a class $\mathcal{R}_d$ with bounded expansion such that $G(n, d/n)$ a.a.s. belongs to $\mathcal{R}_d$.

A consequence

Let $d > 0$. Then for every $p \in \mathbb{N}$ there exists a graph D_p such that $G(n, d/n)$ a.a.s. satisfies

$$C_{2p+1} \longrightarrow G(n, d/n) \qquad \Longleftrightarrow \qquad G(n, d/n) \not\longrightarrow D_p.$$

In other words: for $G(n, d/n)$, each odd-cycle is the "high-probability" obstruction of some H-coloring problem.

Queue number

37 Overview

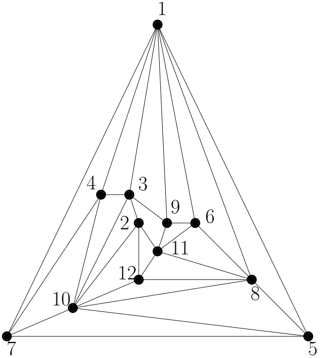
38 Random graphs

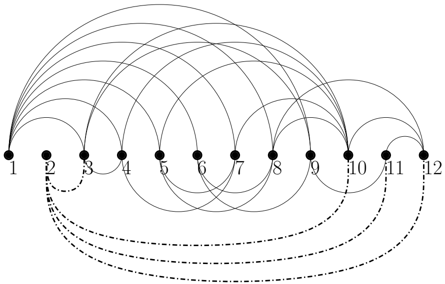
39 Queue number

40 Stack number

41 Non repetitive colorings

Queue layout and queue number





Queue number

Lemma (Dujmović and Wood, 2005)

If some $(\leq t)$-subdivision of a graph G has a k-queue layout, then $\mathrm{qn}(G) \leq \frac{1}{2}(2k+2)^{2t} - 1$, *and if* $t = 1$ *then* $\mathrm{qn}(G) \leq 2k(k+1)$.

Lemma (Heath, 1992; Pemmaraju, 1992; Dujmović, 2004)

Every k-queue graph has average degree less than $4k$.

Queue number

Lemma (Dujmović and Wood, 2005)

If some ($\leq t$)-subdivision of a graph G has a k-queue layout, then $\mathrm{qn}(G) \leq \frac{1}{2}(2k+2)^{2t} - 1$, and if $t = 1$ then $\mathrm{qn}(G) \leq 2k(k+1)$.

Lemma (Heath, 1992; Pemmaraju, 1992; Dujmović, 2004)

Every k-queue graph has average degree less than $4k$.

Queue number

Theorem

Graphs of bounded queue-number have bounded expansion. In particular

$$\widetilde{\nabla}_d(G) < (2k+2)^{4d}$$

for every k-queue graph G.

Proof.

Let $X \in G \,\widetilde{\triangledown}\, d$ and H be the subdivision of X in G. Then $\mathrm{qn}(H) \leq k \;\Rightarrow\; \mathrm{qn}(X) < \frac{1}{2}(2k+2)^{4d}$. Thus the average degree of X is $\leq \delta = 2(2k+2)^{4d}$. Hence $\widetilde{\nabla}_d(G) \leq (2k+2)^{4d}$. $\square$

Queue number (a better bound)

Lemma

Let G be a graph with a k-queue layout and let $H \in G \,\triangledown\, r$. Then H has an $f_r(k)$-queue layout, where

$$f_r(k) := 2k\left(\frac{(2k)^{r+1}-1}{2k-1}\right)^2 .$$

Theorem

If G has a k-queue layout then

$$\nabla_d(G) \leq 8k\left(\frac{(2k)^{d+1}-1}{2k-1}\right)^2 .$$

Queue number (a better bound)

Lemma

Let G be a graph with a k-queue layout and let $H \in G \,\triangledown\, r$. Then H has an $f_r(k)$-queue layout, where

$$f_r(k) := 2k\left(\frac{(2k)^{r+1}-1}{2k-1}\right)^2 .$$

Theorem

If G has a k-queue layout then

$$\triangledown_d(G) \leq 8k\left(\frac{(2k)^{d+1}-1}{2k-1}\right)^2 .$$

Stack number

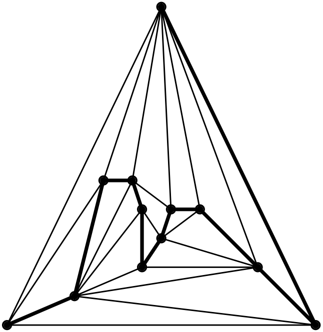
37 Overview

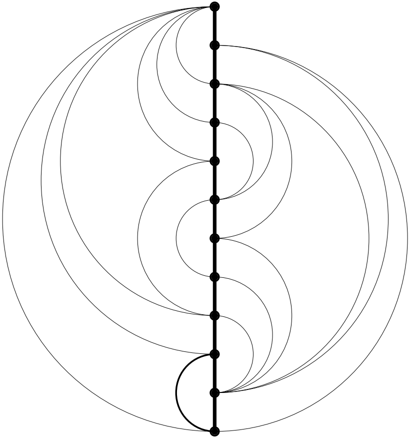
38 Random graphs

39 Queue number

40 Stack number

41 Non repetitive colorings





Lemma (Enomoto, Miyauchi and Ota, 1999)

Let G be a graph such that some ($\leq t$)-subdivision of G has a k-stack layout for some $k \geq 3$. Then

$$\|G\| \leq \frac{4k(5k-5)^{t+1}}{5k-6}|G| \ .$$

Theorem

Graphs of bounded stack number have bounded expansion. In particular:

$$\widetilde{\nabla}_r(G) \leq \frac{4k(5k-5)^{2r+1}}{5k-6}$$

for every k-stack graph G.

Lemma (Enomoto, Miyauchi and Ota, 1999)

Let G be a graph such that some $(\leq t)$-subdivision of G has a k-stack layout for some $k \geq 3$. Then

$$\|G\| \leq \frac{4k(5k-5)^{t+1}}{5k-6}|G| \ .$$

Theorem

Graphs of bounded stack number have bounded expansion. In particular:

$$\widetilde{\nabla}_r(G) \leq \frac{4k(5k-5)^{2r+1}}{5k-6}$$

for every k-stack graph G.

Non repetitive colorings

37 Overview

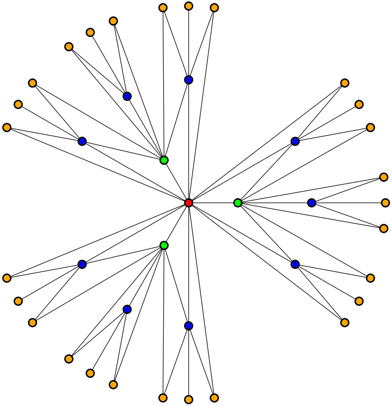
38 Random graphs

39 Queue number

40 Stack number

41 Non repetitive colorings

Thue number



Topological parameters

A real-valued graph parameter α is *strongly topological* if there exist functions f_1, f_2 such that for every graph G and any ≤ 1-subdivision G' of G the following holds:

$$\alpha(G') \leq f_1(\alpha(G)) \qquad \text{and} \qquad \alpha(G) \leq f_2(\alpha(G')).$$

Examples: girth, Hadwiger number, queue number, Thue number

Lemma

Let ϱ be a graph parameter and let $\mathcal{C}$ be an infinite class. Then the two following conditions are equivalent:

- *there exists no positive integer r such that $\varrho(\mathcal{C} \mathbin{\widetilde{\nabla}} r) = \infty$,*
- *there exists a strongly topological monotone graph parameter $\widetilde{\varrho}$ bounding ϱ such that $\widetilde{\varrho}(\mathcal{C}) < \infty$.*

Topological parameters

A real-valued graph parameter α is *strongly topological* if there exist functions f_1, f_2 such that for every graph G and any ≤ 1-subdivision G' of G the following holds:

$$\alpha(G') \leq f_1(\alpha(G)) \qquad \text{and} \qquad \alpha(G) \leq f_2(\alpha(G')).$$

Examples: girth, Hadwiger number, queue number, Thue number

Lemma

Let ϱ be a graph parameter and let $\mathcal{C}$ be an infinite class. Then the two following conditions are equivalent:

- *there exists no positive integer r such that $\varrho(\mathcal{C} \,\widetilde{\nabla}\, r) = \infty$,*
- *there exists a strongly topological monotone graph parameter $\widetilde{\varrho}$ bounding ϱ such that $\widetilde{\varrho}(\mathcal{C}) < \infty$.*

Thue number

Lemma

$\forall G' \leq$ 1-subdivision of G, if $\pi(G') \leq k$, then

$$\pi(G) \leq (k+1) \cdot 2^{2(k+1)^2} ((k+1)(k+2)(2k+3))^{(k+1) \cdot 2^{2(k+1)^2} - 1}.$$

Conversely, $\pi(G') \leq \pi(G) + 1$.

Lemma (Barát and Wood,2008)

For every graph G, $\|G\| \leq (2\pi(G) - 2)|G|$.

Theorem

For every k, the class of all graphs G with $\pi(G) \leq k$ has bounded expansion.

Thue number

Lemma

$\forall G' \leq$ 1-subdivision of G, if $\pi(G') \leq k$, then

$$\pi(G) \leq (k+1) \cdot 2^{2(k+1)^2} ((k+1)(k+2)(2k+3))^{(k+1) \cdot 2^{2(k+1)^2} - 1}.$$

Conversely, $\pi(G') \leq \pi(G) + 1$.

Lemma (Barát and Wood, 2008)

For every graph G, $\|G\| \leq (2\pi(G) - 2)|G|$.

Theorem

For every k, the class of all graphs G with $\pi(G) \leq k$ has bounded expansion.

Thue number

Lemma

$\forall G' \leq$ 1-subdivision of G, if $\pi(G') \leq k$, then

$$\pi(G) \leq (k+1) \cdot 2^{2(k+1)^2} ((k+1)(k+2)(2k+3))^{(k+1)\cdot 2^{2(k+1)^2}-1}.$$

Conversely, $\pi(G') \leq \pi(G) + 1$.

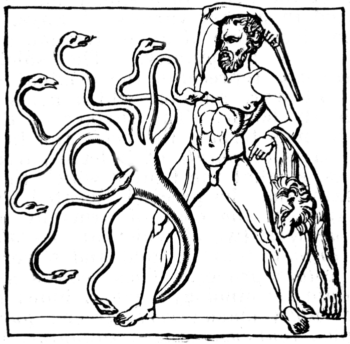
Lemma (Barát and Wood,2008)

For every graph G, $\|G\| \leq (2\pi(G) - 2)|G|$.

Theorem

For every k, the class of all graphs G with $\pi(G) \leq k$ has bounded expansion.

Algorithmic applications



Low tree depth decomposition

42 Low tree depth decomposition

43 Subgraph isomorphism problem

44 First-order decidability

Algorithmic version of LTDD theorem

Procedure A

for $k = 1$ **to** $2^{p-1} + 1$ **do**
  Compute a fraternal augmentation.
end for
Compute depth p transitivity
Compute the conflict graph and color it

Theorem

For every integer p there exists a polynomial P_p (of degree about 2^{2^p}) such that for every graph G Procedure A computes a $(p+1)$-centered coloring of G with $N_p(G) \leq P_p(\widetilde{\nabla}_{2^{p-2}+\frac{1}{2}}(G))$ colors in time $O(N_p(G)n)$-time.

Algorithmic version of LTDD theorem

Procedure A

for $k = 1$ **to** $2^{p-1} + 1$ **do**
  Compute a fraternal augmentation.
end for
Compute depth p transitivity
Compute the conflict graph and color it

Theorem

For every integer p there exists a polynomial P_p (of degree about 2^{2^p}) such that for every graph G Procedure A computes a $(p+1)$-centered coloring of G with $N_p(G) \leq P_p(\widetilde{\nabla}_{2^{p-2}+\frac{1}{2}}(G))$ colors in time $O(N_p(G)n)$-time.

Subgraph isomorphism problem

42 Low tree depth decomposition

43 Subgraph isomorphism problem

44 First-order decidability

Subgraph isomorphism problem		
Context	*Complexity*	*Reference(s)*
General	$O(n^{0.792\ \lvert H \rvert})$	Nešetřil-Poljak using Coppersmith-Winograd
Bounded tree-width	$O(n)$	Eppstein; Courcelle
Planar	$O(n)$	Eppstein
Bounded genus	$O(n)$	Eppstein
Bounded expansion	$O(n)$	POM-Nešetřil
Nowhere dense	$n^{1+o(1)}$	POM-Nešetřil

Theorem

Let $L : V(G) \to 2^{V(F)}$ *be a list assignment. The number of homomorphisms* $f : F \to G$ *such that* $u \in L(f(u))$ *for every* $u \in F$ *can be computed in time* $O(|F| \mathrm{td}(G) 2^{|F| \mathrm{td}(G)} |G|)$.

Corollary

The number of subgraphs of G isomorphic to fixed F can be computed in time $O(|F| \mathrm{td}(G) 2^{|F| \mathrm{td}(G)} |G|)$.

First-order decidability

42 Low tree depth decomposition

43 Subgraph isomorphism problem

44 First-order decidability

First-order decidability

Theorem (Dvořák, Kráľ, Thomas 2010)

Let $\mathcal{C}$ be a class of graphs with bounded expansion, L a language and ϕ an L-sentence. There exists a linear time algorithm that decides whether an L-structure with Gaifman graph $\in \mathcal{C}$ satisfies ϕ.

Theorem (Dvořák, Kráľ, Thomas 2010)

Let $\mathcal{C}$ be a class of graphs with locally bounded expansion, L a language and ϕ an L-sentence. There exists an almost linear time algorithm that decides whether an L-structure with Gaifman graph $\in \mathcal{C}$ satisfies ϕ.

First-order decidability

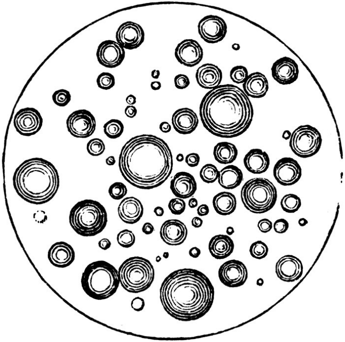
Theorem (Dvořák, Kráľ, Thomas 2010)

Let $\mathcal{C}$ be a class of graphs with bounded expansion, L a language and ϕ an L-sentence. There exists a linear time algorithm that decides whether an L-structure with Gaifman graph $\in \mathcal{C}$ satisfies ϕ.

Theorem (Dvořák, Kráľ, Thomas 2010)

Let $\mathcal{C}$ be a class of graphs with locally bounded expansion, L a language and ϕ an L-sentence. There exists an almost linear time algorithm that decides whether an L-structure with Gaifman graph $\in \mathcal{C}$ satisfies ϕ.

Hyperfiniteness and Separators



Hyperfinite classes

45 Hyperfinite classes

46 Property testing

47 Weakly hyperfinite classes

48 Separators

Hyperfinite classes

Definition (Elek, 2006)

A class $\mathcal{C}$ of (finite) graphs is *hyperfinite* if for every positive real $\epsilon > 0$ there exists a positive integer $K(\epsilon)$ such that every graph $G \in \mathcal{C}$ has a subset of at most $\epsilon|G|$ edges whose deletion leaves no connected component of order greater than $K(\epsilon)$.

Motivation

- Local weak convergence (Benjamini, Schramm; Aldous, Lyons)
- Property testing (Goldreich and Ron model for bounded degree)

Hyperfinite classes

Definition (Elek, 2006)

A class $\mathcal{C}$ of (finite) graphs is *hyperfinite* if for every positive real $\epsilon > 0$ there exists a positive integer $K(\epsilon)$ such that every graph $G \in \mathcal{C}$ has a subset of at most $\epsilon|G|$ edges whose deletion leaves no connected component of order greater than $K(\epsilon)$.

Motivation

- Local weak convergence (Benjamini, Schramm; Aldous, Lyons)
- Property testing (Goldreich and Ron model for bounded degree)

Hyperfiniteness and convergence

- (G, o): rooted graph; $B_G(o, r)$: ball of radius r (isom. type);
- $\hat{\mathcal{G}}_M$: locally finite graphs with $\Delta \le M$; $\hat{\mathcal{G}}^0_M$: finite graphs with $\Delta \le M$;
- $\rho((G, o), (G', o')) = 1/\sup\{r : \ B_G(o, r) \cong B_{G'}(o', r)\}$ (metric);
- $\mathfrak{M}_M$: space of all probability measures on $\hat{\mathcal{G}}_M$ that are measurable with respect to the Borel σ-field of ρ
- *weak convergence* of measures: $\mathrm{E}_n(f) \to \mathrm{E}(f)$ for all bounded, continuous functions $f \rightsquigarrow \mathfrak{M}_M$ compact;
- $\Psi : \hat{\mathcal{G}}^0_M \to \mathfrak{M}_M$: for $G \in \hat{\mathcal{G}}^0_M$, choose randomly o, let $G_o =$ connected component of o; then $\Psi(G) =$ law of (G_o, o).

Hyperfiniteness and convergence

- (G, o): rooted graph; $B_G(o, r)$: ball of radius r (isom. type);
- $\hat{\mathcal{G}}_M$: locally finite graphs with $\Delta \le M$; $\hat{\mathcal{G}}_M^0$: finite graphs with $\Delta \le M$;
- $\rho((G, o), (G', o')) = 1/\sup\{r : B_G(o, r) \cong B_{G'}(o', r)\}$ (metric);
- $\mathfrak{M}_M$: space of all probability measures on $\hat{\mathcal{G}}_M$ that are measurable with respect to the Borel σ-field of ρ
- *weak convergence* of measures: $\mathrm{E}_n(f) \to \mathrm{E}(f)$ for all bounded, continuous functions $f \rightsquigarrow \mathfrak{M}_M$ compact;
- $\Psi : \hat{\mathcal{G}}_M^0 \to \mathfrak{M}_M$: for $G \in \hat{\mathcal{G}}_M^0$, choose randomly o, let $G_o =$ connected component of o; then $\Psi(G) =$ law of (G_o, o).

Hyperfiniteness and convergence

- (G, o): rooted graph; $B_G(o, r)$: ball of radius r (isom. type);
- $\hat{\mathcal{G}}_M$: locally finite graphs with $\Delta \le M$; $\hat{\mathcal{G}}^0_M$: finite graphs with $\Delta \le M$;
- $\rho((G, o), (G', o')) = 1/\sup\{r : B_G(o, r) \cong B_{G'}(o', r)\}$ (metric);
- $\mathfrak{M}_M$: space of all probability measures on $\hat{\mathcal{G}}_M$ that are measurable with respect to the Borel σ-field of ρ
- *weak convergence* of measures: $\mathrm{E}_n(f) \to \mathrm{E}(f)$ for all bounded, continuous functions $f \rightsquigarrow \mathfrak{M}_M$ compact;
- $\Psi : \hat{\mathcal{G}}^0_M \to \mathfrak{M}_M$: for $G \in \hat{\mathcal{G}}^0_M$, choose randomly o, let $G_o =$ connected component of o; then $\Psi(G) =$ law of (G_o, o).

Hyperfiniteness and convergence

- (G, o): rooted graph; $B_G(o, r)$: ball of radius r (isom. type);
- $\hat{\mathcal{G}}_M$: locally finite graphs with $\Delta \leq M$; $\hat{\mathcal{G}}^0_M$: finite graphs with $\Delta \leq M$;
- $\rho((G, o), (G', o')) = 1/\sup\{r : \; B_G(o, r) \cong B_{G'}(o', r)\}$ (metric);
- $\mathfrak{M}_M$: space of all probability measures on $\hat{\mathcal{G}}_M$ that are measurable with respect to the Borel σ-field of ρ
- *weak convergence* of measures: $\mathrm{E}_n(f) \to \mathrm{E}(f)$ for all bounded, continuous functions $f \rightsquigarrow \mathfrak{M}_M$ compact;
- $\Psi : \hat{\mathcal{G}}^0_M \to \mathfrak{M}_M$: for $G \in \hat{\mathcal{G}}^0_M$, choose randomly o, let $G_o =$ connected component of o; then $\Psi(G) =$ law of (G_o, o).

Hyperfiniteness and convergence

- (G, o): rooted graph; $B_G(o, r)$: ball of radius r (isom. type);
- $\hat{\mathcal{G}}_M$: locally finite graphs with $\Delta \le M$; $\hat{\mathcal{G}}^0_M$: finite graphs with $\Delta \le M$;
- $\rho((G, o), (G', o')) = 1/\sup\{r : \; B_G(o, r) \cong B_{G'}(o', r)\}$ (metric);
- $\mathfrak{M}_M$: space of all probability measures on $\hat{\mathcal{G}}_M$ that are measurable with respect to the Borel σ-field of ρ
- *weak convergence* of measures: $\mathrm{E}_n(f) \to \mathrm{E}(f)$ for all bounded, continuous functions f $\rightsquigarrow$ $\mathfrak{M}_M$ compact;
- $\Psi : \hat{\mathcal{G}}^0_M \to \mathfrak{M}_M$: for $G \in \hat{\mathcal{G}}^0_M$, choose randomly o, let $G_o =$ connected component of o; then $\Psi(G) =$ law of (G_o, o).

Hyperfiniteness and convergence

- (G, o): rooted graph; $B_G(o, r)$: ball of radius r (isom. type);
- $\hat{\mathcal{G}}_M$: locally finite graphs with $\Delta \leq M$; $\hat{\mathcal{G}}^0_M$: finite graphs with $\Delta \leq M$;
- $\rho((G, o), (G', o')) = 1/\sup\{r : \ B_G(o, r) \cong B_{G'}(o', r)\}$ (metric);
- $\mathfrak{M}_M$: space of all probability measures on $\hat{\mathcal{G}}_M$ that are measurable with respect to the Borel σ-field of ρ
- *weak convergence* of measures: $\mathrm{E}_n(f) \to \mathrm{E}(f)$ for all bounded, continuous functions $f \rightsquigarrow \mathfrak{M}_M$ compact;
- $\Psi : \hat{\mathcal{G}}^0_M \to \mathfrak{M}_M$: for $G \in \hat{\mathcal{G}}^0_M$, choose randomly o, let $G_o =$ connected component of o; then $\Psi(G) =$ law of (G_o, o).

Property testing

45 Hyperfinite classes

46 Property testing

47 Weakly hyperfinite classes

48 Separators

Property testing

$\mathcal{P}$, a class of graphs (*property*), G a graph with $\Delta(G) \leq d$, $\epsilon > 0$.

- If $G \in \mathcal{P}$, G has the property $\mathcal{P}$;
- if $\geq \epsilon d|G|$ adjacencies have to be changed to make it $\in \mathcal{P}$, then G is ϵ-far from $\mathcal{P}$.

Testing algorithm

A *tester* $\mathcal{T}$ for $\mathcal{P}$ and accuracy ϵ: *randomized* algorithm

- *Input:* $|G|$ and adjacency list of G;
- *Output:* **accept** or **reject**;
  - $G \in \mathcal{P} \Rightarrow$ **accept** with probability $\geq 2/3$;
  - G ϵ-far from $\mathcal{P} \Rightarrow$ **reject** with probability $\geq 2/3$;

Query complexity $q_{\mathcal{T}}(n)$; $\mathcal{P}$ *testable* if $\sup_n q_{\mathcal{T}}(n) < C(\epsilon)$.

Property testing

$\mathcal{P}$, a class of graphs (*property*), G a graph with $\Delta(G) \le d$, $\epsilon > 0$.

- If $G \in \mathcal{P}$, G has the property $\mathcal{P}$;
- if $\ge \epsilon d|G|$ adjacencies have to be changed to make it $\in \mathcal{P}$, then G is ϵ-far from $\mathcal{P}$.

Testing algorithm

A *tester* $\mathcal{T}$ for $\mathcal{P}$ and accuracy ϵ: *randomized* algorithm

- *Input:* $|G|$ and adjacency list of G;
- *Output:* **accept** or **reject**;
  - $G \in \mathcal{P} \Rightarrow$ **accept** with probability $\ge 2/3$;
  - G ϵ-far from $\mathcal{P} \Rightarrow$ **reject** with probability $\ge 2/3$;

Query complexity $q_{\mathcal{T}}(n)$; $\mathcal{P}$ *testable* if $\sup_n q_{\mathcal{T}}(n) < C(\epsilon)$.

Hyperfiniteness and property testing

Theorem (Benjamini, Schramm and Shapira, 2008)

Every monotone hyperfinite graph property is testable.

For instance, planarity is testable in the bounded degree model.

Idea of the proof

Pseudometric $\rho_r(G, G') = \sum_H \left| \frac{\#v,\ B_G(v,r) \cong H}{|G|} - \frac{\#v',\ B_{G'}(v',r) \cong H}{|G'|} \right|$;
If $\exists r$ such that

$$\inf_{G \in \mathcal{P}, G' \in \mathcal{Q}} \rho_r(G, G') > 0$$

then $\mathcal{P}$ and $\mathcal{Q}$ are *distinguishable*.
$\forall \epsilon > 0, \mathcal{P}$ and ϵ-far($\mathcal{P}$) distinguishable $\to \mathcal{P}$ testable.

Weakly hyperfinite classes

45 Hyperfinite classes

46 Property testing

47 Weakly hyperfinite classes

48 Separators

Weakly hyperfinite classes

Definition

$\mathcal{C}$ is *weakly hyperfinite* if $\forall \epsilon > 0, \exists K(\epsilon),\ \forall G \in \mathcal{C},\ G$ has a subset of at most $\epsilon|G|$ vertices whose deletion leaves no connected component of order greater than $K(\epsilon)$.

Lemma

A monotone class $\mathcal{C}$ of graphs with bounded average degree is weakly hyperfinite if and only if for every integer M the class $\mathcal{C} \cap \hat{\mathcal{G}}^0_M$ is hyperfinite.

Weakly hyperfinite classes

Definition

$\mathcal{C}$ is *weakly hyperfinite* if $\forall \epsilon > 0, \exists K(\epsilon)$, $\forall G \in \mathcal{C}$, G has a subset of at most $\epsilon|G|$ vertices whose deletion leaves no connected component of order greater than $K(\epsilon)$.

Lemma

A monotone class $\mathcal{C}$ of graphs with bounded average degree is weakly hyperfinite if and only if for every integer M the class $\mathcal{C} \cap \hat{\mathcal{G}}^0_M$ is hyperfinite.

Separators

45 Hyperfinite classes

46 Property testing

47 Weakly hyperfinite classes

48 Separators

Vertex separators

α-vertex separator of G: subset S such that every connected component of $G - S$ contains at most αn vertices.

$$s_G(i) = \max_{\substack{|A| \le i, \\ A \subseteq V(G)}} \min\{|S| : S \text{ is a } \tfrac{1}{2}\text{-vertex separator of } G[A]\},$$

$$\varsigma(n) = \sup_{G \in \mathcal{C}, |G| \le n} \min\{|S| : S \text{ is a } \tfrac{1}{2}\text{-vertex separator of } G\}.$$

Hence if $\mathcal{C}$ is hereditary,

$$\varsigma(n) = \sup_{G \in \mathcal{C}} s_G(n).$$

Concave approximation

- The *convex conjugate* of a lower semi-continuous function $\phi : X \to \mathbb{R} \cup \{\infty\}$ is the function $\phi^\star : X^\star \to \mathbb{R} \cup \{\infty\}$ defined by
$$\phi^\star(x^\star) = \sup\{\langle x^\star, x\rangle - \phi(x) : x \in X\}.$$
- The *convex biconjugate* $\phi^{\star\star}$ of ϕ is the *closed convex hull* of ϕ, i.e. the largest lower semi-continuous convex function smaller than ϕ.
- for non-decreasing function $f : \mathbb{N} \to \mathbb{R}^+$ define $\hat{f}(x) = -g^{\star\star}(-x)$, where
$$g(x) = f(\lfloor x \rfloor) + (x - \lfloor x \rfloor)(f(\lceil x \rceil) - f(\lfloor x \rfloor)).$$
$\to$ $\hat{f}$ is the smallest upper continuous concave function $\geq f$.

Concave approximation

- The *convex conjugate* of a lower semi-continuous function $\phi : X \to \mathbb{R} \cup \{\infty\}$ is the function $\phi^\star : X^\star \to \mathbb{R} \cup \{\infty\}$ defined by
$$\phi^\star(x^\star) = \sup\{\langle x^\star, x\rangle - \phi(x) : x \in X\}.$$
- The *convex biconjugate* $\phi^{\star\star}$ of ϕ is the *closed convex hull* of ϕ, i.e. the largest lower semi-continuous convex function smaller than ϕ.
- for non-decreasing function $f : \mathbb{N} \to \mathbb{R}^+$ define $\hat{f}(x) = -g^{\star\star}(-x)$, where
$$g(x) = f(\lfloor x \rfloor) + (x - \lfloor x \rfloor)(f(\lceil x \rceil) - f(\lfloor x \rfloor)).$$
$\to$ $\hat{f}$ is the smallest upper continuous concave function $\geq f$.

Concave approximation

- The *convex conjugate* of a lower semi-continuous function $\phi : X \to \mathbb{R} \cup \{\infty\}$ is the function $\phi^\star : X^\star \to \mathbb{R} \cup \{\infty\}$ defined by
$$\phi^\star(x^\star) = \sup\{\langle x^\star, x\rangle - \phi(x) : x \in X\}.$$
- The *convex biconjugate* $\phi^{\star\star}$ of ϕ is the *closed convex hull* of ϕ, i.e. the largest lower semi-continuous convex function smaller than ϕ.
- for non-decreasing function $f : \mathbb{N} \to \mathbb{R}^+$ define $\hat{f}(x) = -g^{\star\star}(-x)$, where
$$g(x) = f(\lfloor x \rfloor) + (x - \lfloor x \rfloor)(f(\lceil x \rceil) - f(\lfloor x \rfloor)).$$

$\rightarrow$ $\hat{f}$ is the smallest upper continuous concave function $\geq f$.

Sublinear separators

Theorem

Let $\mathcal{C}$ be a monotone class of graphs. The following are equivalent:

- *the graphs in $\mathcal{C}$ have sublinear vertex separators:*

$$\limsup_{G\in\mathcal{C}} \frac{\min\{|S| : S \text{ is a } \frac{1}{2}\text{-vertex separator of } G\}}{|G|} = 0;$$

- $\lim_{n\to\infty} \sup_{G\in\mathcal{C}} \frac{s_G(n)}{n} = 0;$ $\limsup_{n\to\infty} \frac{\hat{\varsigma}(n)}{n} = 0;$ $\limsup_{n\to\infty} \frac{\varsigma(n)}{n} = 0;$
- $\limsup_{G\in\mathcal{C}} \frac{\mathrm{tw}(G)}{|G|} = 0;$ $\limsup_{G\in\mathcal{C}} \frac{\mathrm{td}(G)}{|G|} = 0.$

Sublinear separators

Theorem

Let $\mathcal{C}$ be a monotone class of graphs. The following are equivalent:

- *the graphs in $\mathcal{C}$ have sublinear vertex separators:*

$$\limsup_{G\in\mathcal{C}} \frac{\min\{|S| : S \text{ is a } \tfrac{1}{2}\text{-vertex separator of } G\}}{|G|} = 0;$$

- $\lim_{n\to\infty} \sup_{G\in\mathcal{C}} \frac{s_G(n)}{n} = 0;$ $\quad \limsup_{n\to\infty} \frac{\hat{\varsigma}(n)}{n} = 0;$ $\quad \limsup_{n\to\infty} \frac{\varsigma(n)}{n} = 0;$

- $\limsup_{G\in\mathcal{C}} \frac{\mathrm{tw}(G)}{|G|} = 0;$ $\quad \limsup_{G\in\mathcal{C}} \frac{\mathrm{td}(G)}{|G|} = 0.$

Sublinear separators

Theorem

Let $\mathcal{C}$ be a monotone class of graphs. The following are equivalent:

- *the graphs in $\mathcal{C}$ have sublinear vertex separators:*

$$\limsup_{G\in\mathcal{C}} \frac{\min\{|S| : S \text{ is a } \tfrac{1}{2}\text{-vertex separator of } G\}}{|G|} = 0;$$

- $\displaystyle\lim_{n\to\infty} \sup_{G\in\mathcal{C}} \frac{s_G(n)}{n} = 0;$ $\quad\displaystyle\limsup_{n\to\infty} \frac{\hat{\varsigma}(n)}{n} = 0;$ $\quad\displaystyle\limsup_{n\to\infty} \frac{\varsigma(n)}{n} = 0;$
- $\displaystyle\limsup_{G\in\mathcal{C}} \frac{\mathrm{tw}(G)}{|G|} = 0;$ $\quad\displaystyle\limsup_{G\in\mathcal{C}} \frac{\mathrm{td}(G)}{|G|} = 0.$

Weak hyperfiniteness from sublinear separators

Theorem

Let $G \in \mathcal{C}$ have order n, let $\mu : V(G) \to [0,1]$ be a probability measure and let $0 < \iota < 1$ be a positive real.
Then there exists a set C of cardinality at most $3\hat{\varsigma}(2\iota n/3)/\iota$ such that no connected component of $G - C$ has a measure greater than ι.

Corollary

Let $\mathcal{C}$ be a monotone class of graphs with bounded average degree and sublinear vertex separators. Then:

- the class $\mathcal{C}$ is weakly hyperfinite;
- if $\mathcal{C}$ has bounded maximum degree, it is hyperfinite.

Weak hyperfiniteness from sublinear separators

Theorem

Let $G \in \mathcal{C}$ have order n, let $\mu : V(G) \to [0,1]$ be a probability measure and let $0 < \iota < 1$ be a positive real.
Then there exists a set C of cardinality at most $3\hat{\varsigma}(2\iota n/3)/\iota$ such that no connected component of $G - C$ has a measure greater than ι.

Corollary

Let $\mathcal{C}$ be a monotone class of graphs with bounded average degree and sublinear vertex separators. Then:

- the class $\mathcal{C}$ is weakly hyperfinite;
- if $\mathcal{C}$ has bounded maximum degree, it is hyperfinite.

Proof

Let $\epsilon > 0$, $G \in \mathcal{C}$, $n = |G|$ and $\mu(v) = 1/n$ for every $v \in V(G)$.

- As $\lim_{x\to\infty} \hat{\varsigma}(x)/x = 0$,

$$\exists K : \ \hat{\varsigma}(2K/3)/(2K/3) < \epsilon/2.$$

Let $\iota = K/n$.

$\Rightarrow$ $\exists S$ of cardinality at most

$$\frac{3\hat{\varsigma}(2\iota n/3)}{\iota} = \frac{\hat{\varsigma}(2K/3)}{2K/3} n \leq \epsilon n$$

such that no connected component of $G - S$ has a measure greater than $\iota = K/n$, i.e. an order greater than K. $\square$

Sublinear separators

- every planar graph has a $O(\sqrt{n})$ separator (Lipton, Tarjan)
- graphs with genus g have a separator of size $O(\sqrt{gn})$ (Gilbert, Hutchinson, and Tarjan)
- graphs excluding K_h as a minor have a $O(h^{3/2}\sqrt{n})$ separator (Alon, Seymour, and Thomas)

Theorem (Plotkin, Rao, and Smith, 1994)

Given a graph with m edges and n nodes, and integers l and h, there is an $O(mn/l)$ time algorithm that will either produce a K_h-minor of depth at most $l \log n$ or will find a separator of size at most $O(n/l + 4lh^2 \log n)$.

Sublinear separators

- every planar graph has a $O(\sqrt{n})$ separator (Lipton, Tarjan)
- graphs with genus g have a separator of size $O(\sqrt{gn})$ (Gilbert, Hutchinson, and Tarjan)
- graphs excluding K_h as a minor have a $O(h^{3/2}\sqrt{n})$ separator (Alon, Seymour, and Thomas)

Theorem (Plotkin, Rao, and Smith, 1994)

Given a graph with m edges and n nodes, and integers l and h, there is an $O(mn/l)$ time algorithm that will either produce a K_h-minor of depth at most $l \log n$ or will find a separator of size at most $O(n/l + 4lh^2 \log n)$.

Polynomial ω-expansion

Theorem

Let $\mathcal{C}$ be a class of graphs with polynomial ω-expansion, i.e. such that there exists a polynomial P which satisfies

$$\forall i \in \mathbb{N}, \forall G \in \mathcal{C}, \qquad \omega(G \triangledown i) \leq P(i).$$

Then the graphs of order n in $\mathcal{C}$ have separators of size $s(n) = O\left((n \log n)^{1-\frac{1}{2d+2}}\right)$ which may be computed in time $O(ns(n)) = o(n^2)$, where d is the degree of P.

Sub-exponential ω-expansion

Theorem

Let $\mathcal{C}$ be a class of graphs with sub-exponential ω-expansion, i.e. such that

$$\limsup_{i\to\infty} \sup_{G\in\mathcal{C}\,\triangledown\, i} \frac{\log \omega(G)}{i} = 0.$$

Then the graphs of order n in $\mathcal{C}$ have separators of size $s(n) = o(n)$ which may be computed in time $O(ns(n)) = o(n^2)$.

Corollary

Let $\mathcal{P}$ be a monotone class of graphs with sub-exponential ω-expansion.

Then the property $G \in \mathcal{P}$ is testable in the bounded degree model.

Sub-exponential ω-expansion

Theorem

Let $\mathcal{C}$ be a class of graphs with sub-exponential ω-expansion, i.e. such that

$$\limsup_{i\to\infty} \sup_{G\in\mathcal{C}\,\triangledown\, i} \frac{\log \omega(G)}{i} = 0.$$

Then the graphs of order n in $\mathcal{C}$ have separators of size $s(n) = o(n)$ which may be computed in time $O(ns(n)) = o(n^2)$.

Corollary

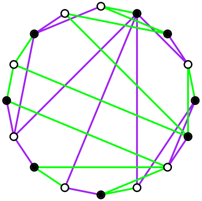
Let $\mathcal{P}$ be a monotone class of graphs with sub-exponential ω-expansion.
Then the property $G \in \mathcal{P}$ is testable in the bounded degree model.

The edge version: Many colors in cycles

Each cycle γ gets ≥ 2 colors:

This means that no cycle is monochromatic.

The minimum number of colors required for G is the *arboricity* $\mathrm{Arb}(G)$.



Theorem (Nash-William, 1964)

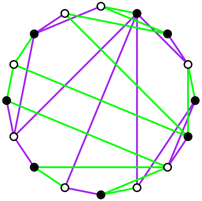
$$\mathrm{Arb}(G) = \max_{H \subseteq G, |H|>1} \left\lceil \frac{\|H\|}{|H|-1} \right\rceil .$$

- Extends to matroids (Edmonds, 1979).

Each cycle γ gets ≥ 2 colors:

This means that no cycle is monochromatic.

The minimum number of colors required for G is the *arboricity* $\mathrm{Arb}(G)$.



Theorem (Nash-William, 1964)

$$\mathrm{Arb}(G) = \max_{H \subseteq G, |H|>1} \left\lceil \frac{\|H\|}{|H|-1} \right\rceil .$$

- Extends to matroids (Edmonds, 1979).

Each cycle γ gets $|\gamma|$ colors:

Proposition

The minimum number of colors required to color a graph G is the maximum number of edges in a 2-connected component (block) of G.

Proof.

$$\begin{aligned} & \text{Each cycle } \gamma \text{ gets } |\gamma| \text{ colors} \\ \iff & \text{every cycle is rainbow} \\ \iff & \text{every 2-connected component is rainbow.} \end{aligned}$$

□

Each cycle γ gets $|\gamma|$ colors:

Proposition

The minimum number of colors required to color a graph G is the maximum number of edges in a 2-connected component (block) of G.

Proof.

$$\begin{aligned} & \text{Each cycle } \gamma \text{ gets } |\gamma| \text{ colors} \\ \iff & \text{every cycle is rainbow} \\ \iff & \text{every 2-connected component is rainbow.} \end{aligned}$$

$\square$

The General Problem

Statement

What is the minimum $N_f(G, p)$ such that the edges of G can be colored by $N_f(G, p)$ colors in such a way that each cycle γ gets

- either $\geq f(|\gamma|)$ colors,
- or $> p$ colors.

- Equivalently, each cycle γ gets $\geq \min(p+1, f(|\gamma|))$ colors;
- For which class $\mathcal{C}$ is it true that

$$\forall p \in \mathbb{N} \qquad \sup_{G \in \mathcal{C}} N_f(G, p) < \infty\,?$$

- Connection with graph *density*?

The General Problem

Statement

What is the minimum $N_f(G,p)$ such that the edges of G can be colored by $N_f(G,p)$ colors in such a way that each cycle γ gets

- either $\geq f(|\gamma|)$ colors,
- or $> p$ colors.

- Equivalently, each cycle γ gets $\geq \min(p+1, f(|\gamma|))$ colors;
- For which class $\mathcal{C}$ is it true that

$$\forall p \in \mathbb{N} \quad \sup_{G \in \mathcal{C}} N_f(G,p) < \infty\,?$$

- Connection with graph *density*?

The General Problem

Statement

What is the minimum $N_f(G,p)$ such that the edges of G can be colored by $N_f(G,p)$ colors in such a way that each cycle γ gets

- either $\geq f(|\gamma|)$ colors,
- or $> p$ colors.

- Equivalently, each cycle γ gets $\geq \min(p+1, f(|\gamma|))$ colors;
- For which class $\mathcal{C}$ is it true that

$$\forall p \in \mathbb{N} \quad \sup_{G \in \mathcal{C}} N_f(G,p) < \infty\,?$$

- Connection with graph *density*?

The General Problem

Statement

What is the minimum $N_f(G,p)$ such that the edges of G can be colored by $N_f(G,p)$ colors in such a way that each cycle γ gets

- either $\geq f(|\gamma|)$ colors,
- or $> p$ colors.

- Equivalently, each cycle γ gets $\geq \min(p+1, f(|\gamma|))$ colors;
- For which class $\mathcal{C}$ is it true that

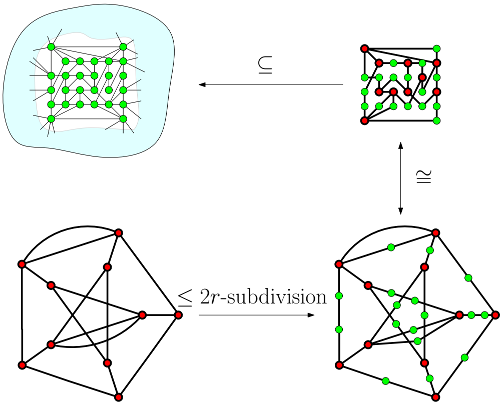
$$\forall p \in \mathbb{N} \quad \sup_{G \in \mathcal{C}} N_f(G,p) < \infty\,?$$

- Connection with graph *density*?

Shallow Topological Multi-Minors – Top Multigrad $\widetilde{\nabla}/_r(G)$

G

$H \in G \,\widetilde{\triangledown}\, r$

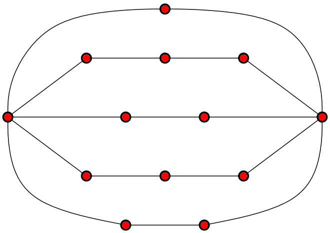


$$\widetilde{\nabla}/_r(G) = \max\left\{\frac{\|H\|}{|H|}, \quad H \in G \,\widetilde{\triangledown}\, r\right\}$$

Relationship between $\widetilde{\nabla\!\!\!/}_r(G)$ and $\widetilde{\nabla}_r(G)$

Let G be a graph and let t be the maximum integer such that G contains two vertices linked by t disjoint paths of length at most $2r+1$. Then

$$\max(t/2, \widetilde{\nabla}_r(G)) \le \widetilde{\nabla\!\!\!/}_r(G) \le (t+1)\widetilde{\nabla}_r(G).$$



Each cycle γ gets $\geq \min(p+1, f(|\gamma|))$ colors:

$f : \mathbb{N} \to \mathbb{N}$ increasing unbounded and $f(x) \leq x$; $p \in \mathbb{N}$.
We denote by $N_f(G, p)$ the number of colors required.

Theorem (Nešetřil, POM, Zhu, 2010)

For every graph G and every integer r,

$$\widetilde{\nabla}_r(G) \leq \mathrm{Poly}_r(f^{-1}(2r+2), N_f(G, 2r+2)).$$

Each cycle γ gets $\geq \min(p+1, f(|\gamma|))$ colors:

$f : \mathbb{N} \to \mathbb{N}$ increasing unbounded and $f(x) \leq x$; $p \in \mathbb{N}$.
We denote by $N_f(G, p)$ the number of colors required.

Sketch of the Proof.

Consider a *good* coloring with $N_f(G, 2r+2)$ colors.

- If H is a 2-connected component of a subgraph induced by $2r+2$ colors then the tree-depth of H is bounded by

$$\mathrm{td}(H) \leq \mathrm{Poly}(f^{-1}(2r+2)).$$

- If all the 2-connected components of subgraphs induced by $2r+2$ colors have tree-depth at most t then

$$\widetilde{\nabla}_r(G) \leq \mathrm{Poly}_r(t, N_f(G, 2r+2)).$$

$\square$

Each cycle γ gets $\geq \min(p+1, f(|\gamma|))$ colors:

Theorem (Nešetřil, POM, Zhu, 2010)

Let $\mathcal{C}$ be a class of graphs. Then the following are equivalent:

1. *There exists increasing unbounded $f : \mathbb{N} \to \mathbb{N}$ such that*

$$\forall p \in \mathbb{N}, \qquad \sup_{G \in \mathcal{C}} N_f(G, p) < \infty$$

2. *Let $f_0(x) = \lceil \log_2 x \rceil$. Then*

$$\forall p \in \mathbb{N}, \qquad \sup_{G \in \mathcal{C}} N_{f_0}(G, p) < \infty$$

3. *the class $\mathcal{C}$ has* bounded expansion, *that is:*

$$\forall r \in \mathbb{N}, \qquad \sup_{G \in \mathcal{C}} \widetilde{\nabla}_r(G) < \infty$$

Each cycle γ gets $\geq \min(p+1, f(|\gamma|))$ colors:

The corollary is *tight*

Let $f_0(x) = \lceil \log_2 x \rceil$ and $f_1(x) = f_0(x) + 1$.
For every integer $p \geq 3$, the minor closed class $\mathcal{C}$ of graphs with tree-depth at most p is such that

$$\sup_{G \in \mathcal{C}} N_{f_0}(G, 2^{p-1}) < \infty \quad \text{but} \quad \sup_{G \in \mathcal{C}} N_{f_1}(G, 2^{p-1}) = \infty.$$

Proof.

By standard pigeonhole argument. □

Each cycle γ gets $\geq \min(p+1, f(|\gamma|))$ colors:

The corollary is *tight*

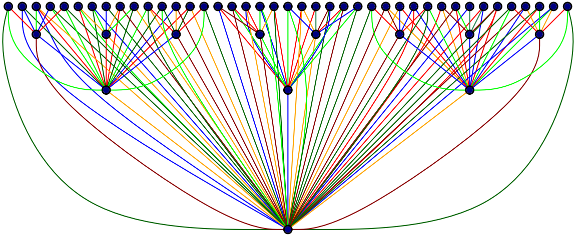
Let $f_0(x) = \lceil \log_2 x \rceil$ and $f_1(x) = f_0(x) + 1$.
For every integer $p \geq 3$, the minor closed class $\mathcal{C}$ of graphs with tree-depth at most p is such that

$$\sup_{G \in \mathcal{C}} N_{f_0}(G, 2^{p-1}) < \infty \quad \text{but} \quad \sup_{G \in \mathcal{C}} N_{f_1}(G, 2^{p-1}) = \infty.$$

Proof.

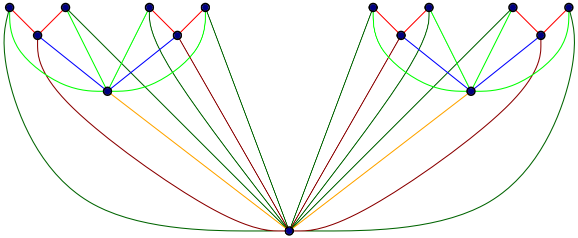
By standard pigeonhole argument. □

The Pigeonhole Argument



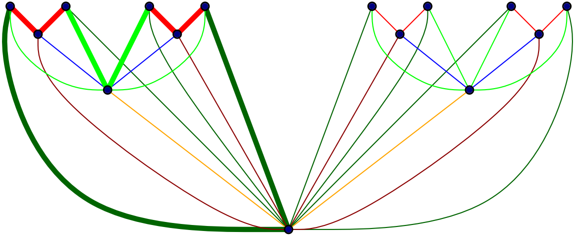
Large graph of tree-depth p with edges colored by N colors

The Pigeonhole Argument



Extract a homogeneous subgraph

The Pigeonhole Argument



Consider a cycle of length 2^{p-1}

Each cycle γ gets $\geq \min(p+1, |\gamma|)$ colors

Definition: for an integer p, the *generalized arboricity* $\mathrm{Arb}_p(G)$ is the minimum number of colors required to color the edges of G is such a way that each cycle γ gets $\geq \min(p+1, |\gamma|)$ colors.

Each cycle γ gets $\geq \min(p+1, |\gamma|)$ colors

Definition: for an integer p, the *generalized arboricity* $\mathrm{Arb}_p(G)$ is the minimum number of colors required to color the edges of G is such a way that each cycle γ gets $\geq \min(p+1, |\gamma|)$ colors.

The *r-acyclic edge chromatic number* $a_r(G)$ of a graph G is the minimum number of colors required to color the edges of G in such a way that adjacent edges receive different colors and every cycle γ receives at least $\min(|\gamma|, r)$ colors (Gerke, Greenhill and Wormald, 2006).

Each cycle γ gets $\geq \min(p+1, |\gamma|)$ colors

Definition: for an integer p, the *generalized arboricity* $\mathrm{Arb}_p(G)$ is the minimum number of colors required to color the edges of G is such a way that each cycle γ gets $\geq \min(p+1, |\gamma|)$ colors.

The *r-acyclic edge chromatic number $a_r(G)$* of a graph G is the minimum number of colors required to color the edges of G in such a way that adjacent edges receive different colors and every cycle γ receives at least $\min(|\gamma|, r)$ colors (Gerke, Greenhill and Wormald, 2006).

For graphs G with maximum degree d, $a_r(G) = \Theta(d^{\lfloor r/2 \rfloor})$ (Greenhill and Pikhurko, 2006).

Each cycle γ gets $\geq \min(p+1, |\gamma|)$ colors

Theorem (Nešetřil, POM, Zhu, 2010)

For an integer p and a multigraph G, it holds:

$$\widetilde{\nabla}_{\frac{p-1}{2}}(G)^{1/p} \leq \mathrm{Arb}_p(G) \leq \mathrm{Poly}_p(\widetilde{\nabla}_{\frac{p-1}{2}}(G))$$

and

$$\max(\widetilde{\nabla}_{\frac{r}{2}}(G)^{1/(r+1)}, \Delta(G)) \leq a_r(G) \leq \mathrm{Poly}_r(\widetilde{\nabla}_{\frac{r}{2}}(G)) + \Delta(G),$$

where $\widetilde{\nabla}_r(G)$ *is the maximum of* $\|H\|/|H|$ *over* **multigraphs** H *such that* G *includes a* $\leq 2r$*-subdivision of* H*.*

Duality

Let $\mathrm{Arb}_p^\star(G)$ be the minimum number of colors required for an edge coloring of G such that every cocyle ω gets $\geq \min(p+1, |\omega|)$ colors.

Arb_p

For every p, n, g, N there exists a graph G such that

$$|G| > n,$$
$$\mathrm{girth}(G) > g,$$
$$\mathrm{Arb}_p(G) > N.$$

$\mathrm{Arb}_p^\star$

Every $(2p+2)$ edge-connected graph G satisfies

$$\mathrm{Arb}_p^\star(G) = p+1.$$

Conclusion

Sparsity

(Graphs, Structures, and Algorithms)

J. Nešetřil and P. Ossona de Mendez

Springer

Thank you!