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II/III

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We start by deriving the Tutte-Berge Formula from the analysis of Edmonds's algorithm we did in the previous lecture. We define $\nu(G)$ to be the size of a maximum matching of the graph G.

Fix a graph G. We want to find a set $U \subseteq V(G)$ such that

$$\nu(G) \leq \frac{1}{2}(|V(G)| + |U| - o(G-U)),$$

where $o(G-U)$ is the number of odd components of $G-U$. Let M be a maximum matching of G given by the algorithm. The last step of the algorithm is of two kinds: either the algorithm did not find any M-alternating path in G, or the algorithm found a blossom B and determined that the maximum size of a matching of G/B is $|M/B|$, so that it has returned M as a maximum matching of G. In the first case, the vertices of G are labelled and the procedure on page 6 of lecture I cannot label vertices anymore. Thus, as we proved, taking for U the set of ODD vertices yields the conclusion. In the second case, we have seen that the formula holds for G/B, with U being the set of ODD vertices, but we still have to find a suitable set U for G. Notice that when B is contracted, the vertex b arose from the contraction is M/B-unmatched, and hence EVEN. As a result, U can be viewed as a subset of $V(G)$. Moreover $o(G-U) = o(G/B-U)$. Indeed, the unique connected component that changes is the component C of $G/B-U$ containing b. As we noted last time, $C = \{b\}$, since b is EVEN. In $G-U$, this component becomes B, and thus is still odd. Now, notice that $|V(G)| = |V(G/B)| + |V(B)| - 1$ and $|M| = |M/B| + \frac{|V(B)|-1}{2}$. Therefore,

$$\begin{aligned}|M| &= \frac{1}{2}(|V(G/B)| + |U| - o(G/B-U)) + \frac{|V(B)|-1}{2}\\ &= \frac{1}{2}(|V(G)| + |U| - o(G/B-U))\\ &= \frac{1}{2}(|V(G)| + |U| - o(G-U)),\end{aligned}$$

as wanted. We have proved the following Tutte-Berge Formula.

Theorem 1. *For every graph G,*

$$\nu(G) = \min_{U \subseteq V(G)} \frac{1}{2}(|V| + |U| - o(G-U)), \tag{1}$$

where $o(G-U)$ is the number of connected components of odd order of $G-U$.

1 A Different Proof of Theorem 1

Let us see a different (and independent) proof of Theorem 1.

Second Proof of Theorem 1. Let G be a graph. We may assume that G is connected (why?). We have already seen in the first lecture that

$$|\nu(G)| \leq \min_{U\subseteq V(G)} \frac{1}{2}(|V(G)| + |U| - o(G-U)) .$$

We prove the reverse inequality by induction on $|V(G)|$, the result holding trivially if $|V(G)| = 1$. Assume now that $|V(G)| \geq 2$ and the formula holds for graphs on at most $|V(G)| - 1$ vertices. We consider two cases.
There is a vertex $v \in V(G)$ that is matched in every maximum matching of G. Set $G' := G - v$. Then, $\nu(G') = \nu(G) - 1$, for otherwise G would have a maximum matching not covering v. Let $U' \subseteq V(G')$ such that $\nu(G') = \frac{1}{2} \cdot (|V(G')| + |U'| - o(G' - U'))$. Further, set $U := U' \cup \{v\}$. Note that $o(G - U) = o(G' - U')$ since $G - U = G' - U'$. Consequently,

$$\begin{aligned}\nu(G) &= \frac{1}{2} \cdot (|V(G')| + |U| - 1 - o(G-U)) + 1 \\ &= \frac{1}{2} \cdot (|V(G)| + |U| - o(G-U)) .\end{aligned}$$

Every vertex of G is unmatched in at least one maximum matching of G. We show that, in this case, $\nu(G) = \frac{1}{2} \cdot (|V(G)| - 1)$. This yields the desired conclusion with $U = \emptyset$. Suppose on the contrary that $\nu(G) < \frac{1}{2} \cdot (|V(G)| - 1)$. For every maximum matching M, we set

$$d(M) := \min\{\mathrm{dist}_G(u, v) : u \text{ and } v \text{ are } M\text{-unmatched}\} ,$$

where dist_G is the distance function in G. We choose M such that $d(M)$ is minimised, and set $d := d(M)$. Note that $d > 1$, for otherwise $M + uv$ would be a matching larger than M. Let t be an internal vertex on a shortest uv-path in G. In particular, t is M-matched. We know that there exists a maximum matching N such that t is N-unmatched (and hence $N \neq M$). Choose such a matching N with moreover the smallest symmetric difference with M.

First, notice that both u and v are N-matched, for otherwise (N, t, u) or (N, t, v) would contradict our choice of (M, u, v). By our assumptions (and since $|M| = |N|$), there exists a vertex $x \neq t$ that is M-matched and N-unmatched (hence $x \notin \{u, v\}$). Let $y \in V$ such that $xy \in M$. Since N is a maximum matching, there exists $z \in V$ such that $yz \in N$. Hence, $z \neq x$. Therefore, $N' := (N \setminus \{yz\}) \cup \{xy\}$ is a maximum matching of G. Moreover, t is N'-unmatched and $|N' \Delta M| < |N \Delta M|$, a contradiction. □

The preceding proof does not give a way to find a set U achieving (1), but, as we have seen, Edmonds's blossom algorithm does. The algorithm also gives us a way to exhibit a very nice structure of graphs, called the Edmonds-Gallai decomposition [3, 4, 5]. The proof of the next theorem can be obtained by analysing a maximum matching output by Edmonds's algorithm, but we omit it. Given a graph G and a set $X \subseteq V(G)$, we define $N(X)$ to be the vertices that do not belong to X and have a neigbhour in X. A graph G is *factor-critical* if for every $v \in V(G)$, the graph $G - v$ has a perfect matching.

Theorem 2. *Let G be a graph and set*

$$\begin{aligned}
&\mathscr{D}(G) := \{v \in V(G) : \exists \text{ a maximum matching of } G \text{ in which } v \text{ is unmatched}\}\,,\\
&\mathscr{A}(G) := N(\mathscr{D}(G))\,,\\
&\mathscr{C}(G) := V(G) \setminus (\mathscr{D}(G) \cup \mathscr{A}(G))\,.
\end{aligned}$$

Then,

1. *The set $\mathscr{A}(G)$ achieves equality in* (1)*;*
2. *$\mathscr{C}(G)$ is the union of the even components of $G - \mathscr{A}(G)$;*
3. *$\mathscr{D}(G)$ is the union of the odd components of $G - \mathscr{A}(G)$; and*
4. *every odd component of $G - \mathscr{A}(G)$ is factor-critical.*

2 Perfect Matchings

A matching M of a graph G is *perfect* if $|M| = |V(G)|/2$, i.e. every vertex of G is incident to an edge in M. Tutte's perfect matching theorem is a direct consequence of Theorem 1.

Theorem 3. *A graph G has a perfect matching if and only if $G - U$ has at most $|U|$ odd components for every $U \subseteq V(G)$.*

As an application of Theorem 3, let us derive the celebrated (and useful) Petersen's theorem [7], proved in 1891. A graph is *cubic* if every vertex has degree 3. Given a graph G, a *bridge* is an edge the removal of which disconnects G.

Theorem 4. *Every cubic bridgeless graph has a perfect matching.*

Proof. By Theorem 3, it suffices to show that $G - U$ has at most $|U|$ odd components, for every $U \subseteq V(G)$. Fix $U \subseteq V(G)$, and let $\mathscr{C}$ be the collection of connected components of $V(G)$. We define $e(U, \mathscr{C})$ to be the number of edges with exactly one end-vertex in U. Since G is cubic,

$$e(U, \mathscr{C}) \leq 3 \cdot |U|\,. \tag{2}$$

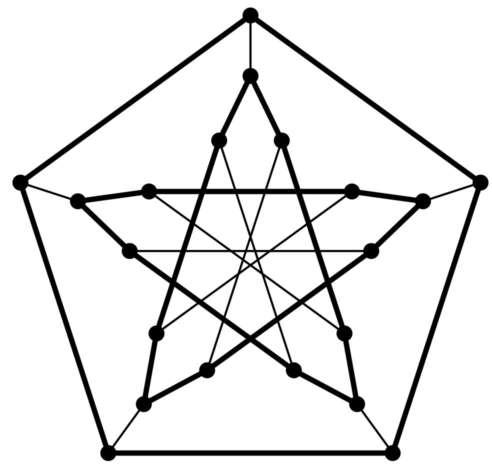

Figure 1: The edges can be partitioned into a 2-factor (in bold) and a perfect matching. The 2-factor is composed of a 5-cycle and a 15-cycle.

On the other hand, every connected component $C \in \mathscr{C}$ is linked to U by at least two edges, since G is bridgeless. Now, let C be an odd component. Let $e(C)$ and $e(U, C)$ be the number of edges with both end-vertices in C and the number of edges with exactly one end-vertex in C, respectively. Since G is cubic,

$$3 \cdot |V(C)| = 2 \cdot e(C) + e(U, C) \,. \tag{3}$$

Observe that the left-hand side of (3) is odd (since C is an odd component). Consequently, $e(U, C)$ is odd, too. Since $e(U, C) \geq 2$, we deduce that $e(U, C) \geq 3$. Therefore, letting $\mathscr{E}$ be the set of odd components, we obtain

$$e(U, \mathscr{C}) \geq 3 \cdot |\mathscr{E}| \,.$$

Combining this with (2), we obtain $|\mathscr{E}| \leq |U|$, as wanted. □

We end this section with some remarks. The first one is an exercise: find a cubic graph without a perfect matching (hence, with bridges!). Can you find one with exactly one bridge? With exactly two?

The second remark concerns the structure of bridgeless cubic graphs. Let G be such a graph. Theorem 4 ensures that G has a perfect matching M. Consider $G - M$: it is a spanning 2-regular subgraph of G, i.e. a collection of vertex-disjoint cycles such that each vertex of G belongs to a cycle. A subgraph that is spanning and k-regular is a *k-factor*. So, a 1-factor is precisely a perfect matching, and $G - M$ is a 2-factor. Theorem 4 yields that every cubic bridgeless graph contains a 1-factor and a 2-factor that are edge-disjoint. See Figure 1.

Let us define the notion of edge-colouring. Given a graph G, a *k-edge-colouring* is a function $c : E(G) \to \{1, \ldots, k\}$ such that $c(e) \neq c(e')$ for every two adjacent edges e and e' of G. The *chromatic index* $\chi'(G)$ of G is the smallest integer k for

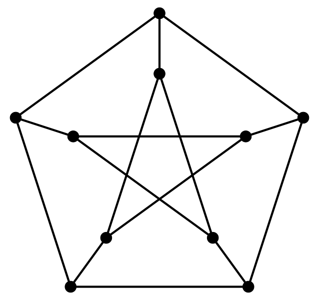

Figure 2: The Petersen graph has chromatic index 4.

which G has a k-edge-colouring. Thus, a k-edge colouring is a partition of the edges into k matchings (each *colour class*, i.e. all the edges of a given colour, is a matching of G). Note that the chromatic index of a graph G is always at least its maximum degree $\Delta(G)$. A celebrated theorem of Vizing [8, 9] ensures that $\chi'(G) \leq \Delta(G) + 1$ for every graph G. Deciding between the two values is NP-complete, even for cubic graphs [6]. More can be said for special classes of graphs, e.g. if G is bipartite then $\chi'(G) = \Delta(G)$. This is not true for cubic graphs, as shown by the Petersen graph (see Figure 2). Cubic graphs with chromatic index index 4 are called *snarks*, because they are hard to find (often, other properties are required to avoid trivialities). Edge-colouring of cubic graphs is linked to deep problems such as the 4-Color Theorem. A *vertex colouring* of a graph is an assignment of colours to the vertices such that no two adjacent vertices have the same colour. The 4-Colour Conjecture states that every planar graph can be vertex-coloured with 4 colours. This conjecture was proved to be true by Appel and Haken [1, 2] in 1977. A century ago, Tait proved that the 4-Colour Conjecture was true if and only if every planar cubic bridgeless graph is 3-edge-colourable. More about edge-colouring in the exercises!

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