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J.-S. Sereni *Matchings in Graphs* *11/2009*

Some Definitions

We just give the definitions and characterisations needed. More background can be found in the treatise by Schrijver [4, Chapter 5].

A *convex combination* of the vectors $x_1, \ldots, x_k \in \mathbf{R}^n$ is a vector equal to $\sum_{i=1}^{k} \lambda_i x_i$ where $\lambda_1, \ldots, \lambda_k \in \mathbf{R}^+$ with $\sum_{i=1}^{k} \lambda_i = 1$. A set $C \subseteq \mathbf{R}^n$ is *convex* if for every $x, x' \in C$ and every $\lambda \in [0,1]$, it holds that $\lambda x + (1-\lambda)y \in C$. The *convex hull* $\operatorname{conv.hull}(X)$ of a set $X \subseteq \mathbf{R}^n$ is the smallest convex set that contains X, that is

$$\left\{ \lambda_1 x_1 + \ldots + \lambda_n x_n : n \in \mathbf{N},\ \forall i \in \{1, \ldots, n\}, x_i \in X \text{ and } \lambda_i \in \mathbf{R}^+,\ \sum_i \lambda_i = 1 \right\}.$$

A *polytope* is the convex hull of finitely many vectors of $\mathbf{R}^n$. A *polyhedron* is a set $X \subseteq \mathbf{R}^n$ such that there exists an $m \times n$-matrix A and a vector $b \in \mathbf{R}^m$ (for some integer m) with

$$X = \{x \in \mathbf{R}^n : Ax \leq b\}.$$

A classical and fundamental result states that a set is a polytope if and only if it is a bounded polyhedron.

An element x of a polyhedron P is a *vertex* if it cannot be expressed as a convex combination of elements in $P \setminus \{x\}$. Let us give a useful characterisation of vertices of polytopes. Let $P = \{x \in \mathbf{R}^n : Ax \leq b\}$ be a polytope, and let $x \in P$. Then, x is a vertex of P if and only if $\operatorname{rank}(A_x) = n$, where A_x is the submatrix of A composed of the row vectors r_i of A such that $r_i x = b_i$.

It follows from this characterisation that if A and b are *rational* (that is, have rational coordinates), then every vertex of P is rational.

1 The (Perfect) Matching Polytope

Let $G = (V, E)$ be a graph. For $U \subseteq V$, we define $\delta_G(U)$ (or just $\delta(U)$) to be the set of edges of G with exactly one end-vertex in U. For every vertex $v \in V$, we set $\delta(v) := \delta(\{v\})$. We identify a matching M with its *incidence vector* $x_M \in \{0,1\}^{|E|}$, where $x_M(e) = 1$ if and only if $e \in M$. For $W \subseteq E$, let $x(W) := \sum_{e \in W} x(e)$.

The *matching polytope* of G is the set

$$\mathscr{M} := \text{conv. hull}\{x_M : M \text{ matching of } G\}\,.$$

Let

$$\mathscr{N} := \left\{x \in [0,1]^{|E|} : \forall v \in V, \quad x(\delta(v)) \le 1\right\}\,.$$

Remark 1.

- The set $\mathscr{N}$ is a polytope (and hence convex).
- $\mathscr{M} \subseteq \mathscr{N}$, since $\mathscr{N}$ is convex and every vertex of $\mathscr{M}$ belongs to $\mathscr{N}$.

In general, $\mathscr{M}$ is not equal to $\mathscr{N}$. For instance, if $G = K_3$ then $(1/2, 1/2, 1/2) \in \mathscr{N} \setminus \mathscr{M}$. However, equality holds if G is bipartite, as we show next.

Theorem 1. *If G is bipartite then $\mathscr{M} = \mathscr{N}$.*

Proof. We know that $\mathscr{M} \subseteq \mathscr{N}$. Suppose that $\mathscr{N} \not\subseteq \mathscr{M}$. Then, it follows from Remark 1 that $\mathscr{N}$ has a non-integral vertex (indeed, every integral point of $\mathscr{N}$ is in $\mathscr{M}$, so if every vertex of $\mathscr{N}$ were integral then $\mathscr{N}$ would be contained in $\mathscr{M}$ by convexity).

By the definition, a vertex cannot be expressed as a convex combination of the other points. Thus, to obtain a contradiction, if suffices to show that every non-integral point of $\mathscr{N}$ is a convex combination of other points of $\mathscr{N}$.

Let x be a non-integral point of $\mathscr{N}$. Let G' be the subgraph of G spanned by the edges e such that $x(e) \notin \{0,1\}$. We apply a standard shifting technique. For the ease of exposition, we consider two cases, regarding whether G' is acyclic.

Suppose that G' contains a cycle. Let $C := v_1e_1v_2e_2 \dots v_ke_k$. Note that k is even since G is bipartite. Set $a := \min\{e_i : 1 \le i \le k\}$ and $b := \max\{e_i : 1 \le i \le k\}$. Further, set $\varepsilon := \min\{a, 1-b\}$. Thus, $\varepsilon > 0$.

We define the point x^- by

$$\begin{aligned} x^-(e) &:= x(e) - \varepsilon \quad \text{if } e = e_{2i+1} \\ x^-(e) &:= x(e) + \varepsilon \quad \text{if } e = e_{2i} \\ x^-(e) &:= x(e) \quad \text{otherwise.} \end{aligned}$$

Then, $x^- \in [0,1]^E \setminus \{x\}$ and for every vertex $v \in V$,

$$\sum_{e\in\delta(v)} x^-(e) = \sum_{e\in\delta(v)} x(e) \le 1\,.$$

Hence, $x^- \in \mathscr{N} \setminus \{x\}$. Similarly, the point x^+ defined by

$$\begin{aligned} x^+(e) &:= x(e) + \varepsilon \quad \text{if } e = e_{2i+1} \\ x^+(e) &:= x(e) - \varepsilon \quad \text{if } e = e_{2i} \\ x^+(e) &:= x(e) \quad \text{otherwise} \end{aligned}$$

belongs to $\mathscr{N} \setminus \{x\}$. Furthermore, $x = \frac{1}{2}(x^- + x^+)$.
The graph G' is acyclic. We apply the same argument to a path of G'. Let $v_1 e_1 \dots v_k$ be a path of G' forming a connected component. Thus, if e is an edge incident with v_1 and distinct from e_1, then $x(e) = 0$. Similarly for v_k. As a result, the exact same definition of x^+ and x^- yield the conclusion. □

1.1 The General Case

We turn our attention to the non-bipartite case. To this end, we first prove a characterisation of the perfect matching polytope, and next we show that it implies a characterisation of the matching polytope, first found be Edmonds [2]. The *perfect matching polytope* of a graph G is the set

$$\mathscr{P} := \text{conv.\,hull}\,\{x_M : \, M \text{ perfect matching of } G\}\ .$$

A characterisation of the *perfect* matching polytope of bipartite graphs is obtained by replacing the inequality by an equality in the definition of $\mathscr{N}$.

Theorem 2. *The perfect matching polytope of a graph G is the set of all $x \in \mathbf{R}^{|E(G)|}$ that satisfy*

$$x(e) \geq 0 \quad \textit{for each } e \in E(G), \tag{1a}$$
$$x(\delta(v)) = 1 \quad \textit{for each } v \in V(G), \tag{1b}$$
$$x(\delta(U)) \geq 1 \quad \textit{for each } U \subseteq V(G) \textit{ with } |U| \textit{ odd.} \tag{1c}$$

This theorem can be proved using Edmonds's algorithm. We provide a direct proof, originally found by Aráoz, Cunningham, Edmonds and Green-Krótki [1] and Schrijver [3]. Consult the book by Schrijver [5, p. 438] for more details.

Proof of Theorem 2. Let $\mathscr{Q}$ be the set of vectors determined by (1). Note that $\mathscr{P} \subseteq \mathscr{Q}$. Assume that the statement of the theorem is not true, and choose a graph G with $\mathscr{Q} \neq \mathscr{P}$ such that $|V(G)| + |E(G)|$ is as small as possible. In particular, G is connected and $|V(G)|$ is even for otherwise $\mathscr{Q} = \emptyset = \mathscr{P}$. Let x be a vertex of $\mathscr{Q}$ that is not in $\mathscr{P}$. Then, $x(e) \in (0, 1)$ for every edge $e \in E(G)$. Indeed, if $x(e) = 0$ then we could remove e, and if $x(e) = 1$ we could remove the end-vertices of e: in either case, this would contradict the minimality of $|V(G)|+|E(G)|$. Thus, G has minimum degree at least 2, so $|E(G)| \geq |V(G)|$. If $|E(G)| = |V(G)|$, then G is a cycle and $\mathscr{Q} = \mathscr{P}$. So, $|E(G)| > |V(G)|$.

Since x is a vertex, it satisfies $|E(G)|$ linearly independent inequalities of (1) with equality, as we pointed out. Since no equality of type (1a) is an equality for x, and there are $|V(G)| < |E(G)|$ inequalities of type (1b), we deduce that x satisfies an inequality of type (1c) with equality. Therefore, there is a set U_1 of odd size such that $x(\delta(U_1)) = 1$ and $|U_1| \in \{3, \ldots, |V(G)| - 3\}$ (why? If $|U_1| \in \{1, |V(G)| - 1\}$, then compare the equality obtained with one of type (1b)...).

We define G_1 to be the graph obtained from G by contracting U_1 into a single vertex u (removing loops, but keeping parallel edges if any). We set $U_2 := V(G) \setminus U_1$ and we define G_2 to be the graph obtained from G by contracting U_2. Note that $|U_2|$ is odd. Let x^i be the projection of x on the edges of G_i, for $i \in \{1, 2\}$.

First, note that x^i satisfies (1) in G_i. Indeed, let X be a subset of $V(G_i)$ of odd order. Let u_i be the vertex arose from the contraction of U_i. If $u_i \notin X$, then the conclusion holds. If $u_i \in X$, then let Y be the subset of $V(G)$ obtained from X by replacing u_i with the vertices of U_i. Since $|U_i|$ is odd, $|Y|$ is odd and therefore $\sum_{e\in\delta(Y)} x(e) \geq 1$, which yields the conclusion since $\sum_{e\in\delta(X)} x^i(e) = \sum_{e\in\delta(Y)} x(e)$.

Consequently, the minimality of G implies that x^i belongs to the perfect matching polytope of G_i. Thus x^i can be expressed as a convex combination of vertices of G_i. Furthermore, x^i is rational since x is. Hence, we infer the existence of (not necessarily distinct) perfect matchings $M_1^i, \ldots, M_k^i$ of G_i such that

$$x^1 = \frac{1}{k}\sum_{j=1}^{k} x_{M_j^1} \text{ and } x^2 = \frac{1}{k}\sum_{j=1}^{k} x_{M_j^2}.$$

For each edge $e \in \delta(U_1)$, we have $x^1(e) = x(e) = x^2(e)$. Thus, the number of indices $j \in \{1, \ldots, k\}$ such that $e \in M_j^1$ is $k \cdot x(e)$. Similarly, the number of indices j such that $e \in M_j^2$ is $k \cdot x(e)$. Therefore, we may assume (up to renumbering the matchings) that for every $j \in \{1, \ldots, k\}$, the perfect matchings M_j^1 and M_j^2 have an edge in $\delta(U_1)$ in common. As a result, the set $M_j := M_j^1 \cup M_j^2$ is a perfect matching of G. Moreover, note that

$$x = \frac{1}{k}\sum_{i=1}^{k} x_{M_j}.$$

Hence, $x \in \mathscr{P}$; a contradiction. $\square$

Theorem 2 allows us to determine also the matching polytope of a graph (Edmonds's matching polytope theorem).

Theorem 3. *The matching polytope of the graph G is determined by*

$$x(e) \geq 0 \quad \text{for each } e \in E(G), \tag{2a}$$
$$x(\delta(v)) \leq 1 \quad \text{for each } v \in V(G), \tag{2b}$$
$$x(E[U]) \leq \left\lfloor \frac{1}{2}|U| \right\rfloor \quad \text{for each } U \subseteq V(G) \text{ with } |U| \text{ odd,} \tag{2c}$$

where $E[U] := E(G[U])$, the set of edges of G with both endvertices in U.

Proof. Every vector of the matching polytope of G satisfies (2). Conversely, let x be a vector satisfying (2). Let G_1 and G_2 be two copies of G. For convenience, a vertex $v_i \in V(G_i)$ is assumed to be the copy of the vertex $v \in V(G)$. For each $v \in V(G)$, add an edge between v_1 and v_2. Let H be the obtained graph.

For each edge e with both endvertices in some G_i, we set $y(e) := x(e)$. For each edge v_1v_2 with $v_i \in V(G_i)$, we set $y(v_1v_2) := 1 - x(\delta(v))$. It suffices to prove that y belongs to the perfect matching polytope of H. Indeed, y would then be a convex combination of perfect matchings of H, which would imply that x is a convex combination of matchings of G (why?).

We use Theorem 2 to prove that y belongs to the perfect matching polytope of H. The vector y satisfies (1a) and (1b) by the definition and because x satisfies (2b). It remains to check that (1c) is satisfied.

Let $U \subseteq V(H)$ with $|U|$ odd. Let us write $U = X_1 \cup Y_2$ with $X, Y \subseteq V(G)$. Observe that $y(\delta_H(U)) \geq y(\delta_H(X_1 \setminus Y_1)) + y(\delta_H(Y_2 \setminus X_2))$ (why?). Consequently, it suffices to prove the result when X and Y are disjoint. Moreover, since $|U|$ is odd, we may assume that $|X|$ is odd, and hence $Y = \emptyset$. Since y satisfies (1b),

$$|X| = \sum_{v \in X} y(\delta_H(v)) = y(\delta_H(X)) + 2 \cdot y(E_H[U])\,,$$

where $E_H[U]$ is the set of edges of H with exactly one endvertex in U. Consequently, as x satisfies (2c), we deduce that

$$y(\delta_H(X)) = |X| - 2 \cdot y(E_H[U]) \geq |X| - 2 \cdot \left\lfloor \frac{1}{2}|X| \right\rfloor = 1\,.$$

Therefore, y belongs to the perfect matching polytope of H by Theorem 1, which concludes the proof. □

2 An Application to Cubic Graphs

Let us see an example of use of the perfect matching polytope. The elegant proof of the following result is due to D. Král' (personal communication).

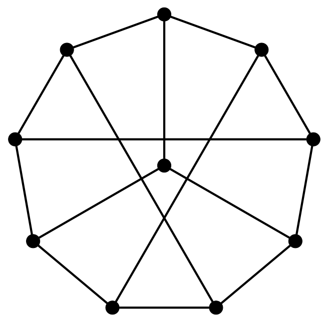

Figure 1: The unique cubic bridgeless graph all of which 2-factors are composed only of 5-cycles is the Petersen graph.

Proposition 4. *A cubic bridgeless graph G has a 2-factor containing a cycle of length different from 5 if and only if G is not the Petersen graph.*

Proof. Let us prove that if all 2-factors of G are composed of 5-cycles only, then G is the Petersen graph. Let $x := (1/3, \dots, 1/3) \in [0,1]^{|E|}$. Since G is cubic and bridgeless, $x \in \mathscr{P}$. Thus, there exist k perfect matchings $M_1, \dots, M_k$ of G and k positive rational numbers $\alpha_1, \dots, \alpha_k$ with $\sum_{i=1}^k \alpha_i = 1$ such that

$$x = \sum_{i=1}^{k} \alpha_i x_{M_i} \, .$$

We randomly choose a perfect matching M among $M_1, \dots, M_k$, the probability that M_i is chosen being α_i for $i \in \{1, \dots, k\}$. Let F be the complement of M and C a 5-cycle of G. What is the probability that C is a cycle of F? We assert that $p := \mathbf{Pr}(E(C) \subseteq F) \le \frac{1}{6}$. To see this, set $X := \delta(V(C))$ and let $w_i := x_{M_i}(X)$. Note that $w_i \ge 1$ for every $i \in \{1, 2, \dots, k\}$ since $|V(C)|$ is odd. Moreover, $E(C) \subseteq F_i$ if and only if $w_i = 5$. Hence, setting $I := \{i \in \{1, \dots, k\} : w_i = 5\}$, we have $p = \sum_{i \in I} \alpha_i$. Now, observe that $\sum_{i=1}^k \alpha_i w_i = x(X) = \frac{5}{3}$. Therefore,

$$\begin{aligned} \frac{5}{3} &= \sum_{i \in I} \alpha_i w_i + \sum_{i \notin I} \alpha_i w_i \\ &\ge \sum_{i \in I} 5 \cdot \alpha_i + \sum_{i \notin I} \alpha_i \\ &= 5p + (1-p) \, , \end{aligned}$$

from which it follows that $p \le \frac{1}{6}$, as asserted.

Let $n := |V(G)|$. If each of the 2-factors F_i is composed of $n/5$ cycles of length 5, then the expected number of 5-cycles in F is $n/5$. Since this expected number is at most the total number of 5-cycles times $1/6$, we deduce that the total number of 5-cycles in G is at least $6n/5$.

Let $v \in V(G)$. Since G is a cubic graph, there are at most six edges at distance two from v. Moreover, each such edge belongs to at most one 5-cycle containing v. As a result, every vertex of G is contained in at most (and hence, exactly) six 5-cycles, and no vertex has an edge at distance 3. Therefore, G has 10 vertices. We also infer that G has no 3- or 4-cycle. Consequently, a short analysis shows that G is the Petersen graph.

Conversely, one can check that the Petersen graph has no 2-factor containing a cycle of length different from 5. $\square$

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