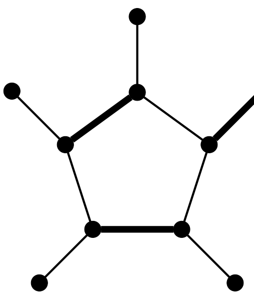
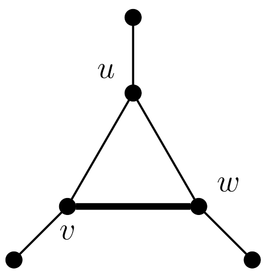
EXERCISES

MASTER PARISIEN DE RECHERCHE EN INFORMATIQUE 2-29-1

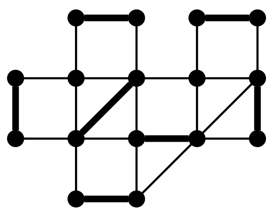
1. Apply Edmonds's blossom algorithm on each of the graphs below, starting with the matching indicated by the bold edges. Indicate the order in which the vertices/edges are processed. For G_2, the order is imposed: start with the vertex u and the edges uv and next uw.



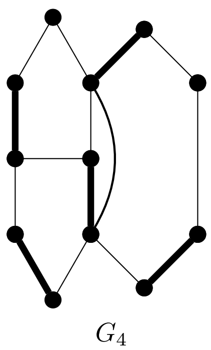
G_1



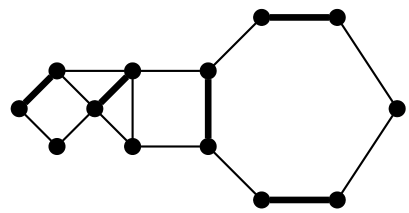
G_2



G_3



G_4



G_5

2. Hall's Theorem reads as follows. *A bipartite graph with bi-partition (A, B) has a matching covering all the vertices of A if and only if*

$$\forall X \subseteq A, \quad |X| \leq |N(X)|,$$

where $N(X)$ is the set of vertices adjacent to a vertex of X.

(i) Prove that every k-regular bipartite graph has a perfect matching.

(ii) Prove that every (non-edgeless) bipartite graph has a matching covering every vertex of maximum degree (use induction on the number of edges, and Hall's Theorem in an auxiliary graph).

(iii) Prove that the chromatic index of a bipartite graph G is equal to the maximum degree of G.

3. A square matrix is *doubly stochastic* if its entries are non-negative real numbers and each row- and column-sums are 1. The *permanent* of the square matrix $A = (a_{i,j})$ is

$$\sum_{\sigma \in \mathscr{S}_n} \prod_{i=1}^{n} a_{i,\sigma(i)} \,.$$

Prove that the permanent of a doubly stochastic matrix is (strictly) positive.

4. An *edge-cover* of a graph G is a set X of edges of G such that every vertex is incident to an edge in X. Let $\rho(G)$ be the minimum number of edges in an edge-cover of G.

(i) Show that a minimum edge-cover consists of a disjoint union of stars (a *star* is a tree having a universal vertex).
(ii) Show that if G is a graph without isolated vertices (that is, without vertices of degree 0), then $\nu(G) + \rho(G) = |V(G)|$.

5. Using induction on $k \geq 1$, build a graph with minimum degree at least k and exactly one perfect matching.
Prove that if G has no perfect matching, then G has a vertex all of which incident edges belong to some maximum matching of G.

6.

(i) Construct a cubic graph with no perfect matching.
(ii) Prove that every cubic graph with at most two bridges has a perfect matching.

7. Let G be a cubic graph, and let $T := xyz$ be a triangle of G (that is, $x, y, z \in V(G)$ and $xy, yz, zx \in E(G)$). Let G/T be the graph obtained from G by contracting T into a single vertex (and keeping parallel edges that may arise).

(i) Prove that G is 3-edge-colourable if and only if G/T is.
(ii) Prove that G is bridgeless if and only if G/T is.

8. Assume that G is a cubic bridgeless graph with no triangles. Let $Q := v_1v_2v_3v_4$ be a 4-cycle of G. Let u_i be the third neighbour of v_i. Let H be the subgraph of G obtained by removing the vertices v_i. Now, let G_1, G_2 and G_3 be the (multi-)graphs obtained from H as depicted in Figure 1. Prove that (at least) one of G_1, G_2 and G_3 is bridgeless.

9. An *Euler trail* of a graph is a trail (that is, a non-elementary path) going through each edge exactly once.

(i) Prove that a graph has an Euler trail if and only if all its vertices have even degree.
(ii) Prove that every connected $2k$-regular graph with an even number of edges has a k-factor.

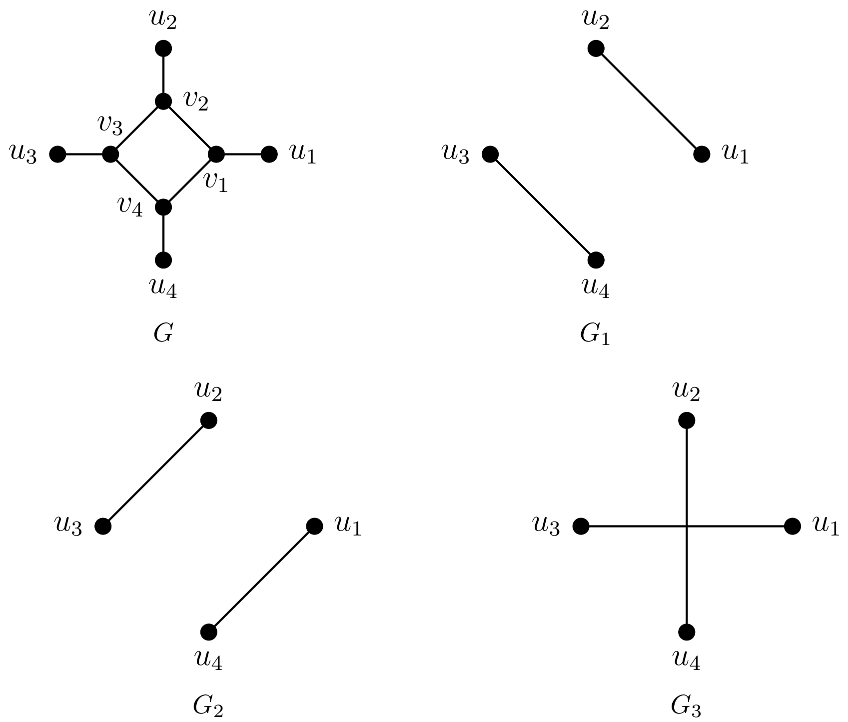


FIGURE 1. The graphs G_1, G_2 and G_3.

10. Show that every $2k$-regular graph is the union of k 2-factors.

11. Prove that a connected graph G has a spanning subgraph all of which vertices have odd degree if and only if $|V(G)|$ is even.

12. Let G be a bridgeless graph with minimum degree at least 3.

(i) Suppose that $v \in V(G)$ has degree at least 4. Show that there exist two edges e and e' both incident to x such that $G - e - e'$ is connected.

(ii) Prove that G has a spanning subgraph with all vertices having a positive and even degree. [Hint: reduce this to the case of cubic graphs, using the following transformation: take $e = xy, e' = xz$ as in (i), remove e, e' and add a new vertex linked precisely to x, y and z.]

13. Let G be a cubic bridgeless graph.

(i) Prove that the vector $\left(\frac{1}{3}, \ldots, \frac{1}{3}\right) \in [0,1]^{|E(G)|}$ belongs to the perfect matching polytope of G.

(ii) Prove that for each $e \in E(G)$, there exists a perfect matching of G that contains e.

(iii) Prove the stronger fact that for every two edges $e, e' \in E(G)$, there exists a perfect matching of G that avoids both e and e'.