Grafy a počty - NDMI078		March 2009
	Lecture 1	
M. Loebl		*J.-S. Sereni*

1 Introduction

Perhaps the first theorem of graph theory is Euler's theorem, and it is also about walking. We first write down an observation, which may be called the *greedy principle of walking*. A graph is *even* if the degree of every vertex is even. A directed graph is *even* if for each vertex, the in-degree equals the out-degree. Given a (directed) graph G, a set of edges of G is *even* if the (directed) subgraph spanned by those edges is even.

Observation 1. *Each even set of (directed) edges may be partitioned into disjoint (directed) cycles.*

Proof. First, note that each non-empty even set contains a cycle since any (directed) path in it may be prolonged. Further, if we delete a (directed) cycle from an even set, we again obtain an even set and we can continue in this way until the remaining set is empty. This yields a desired partition. □

A (directed) graph is *Eulerian* if it has an *Euler tour*, i.e. a closed (directed) trail containing all the (directed) edges. Here comes Euler's theorem.

Theorem 1. *A (directed) graph is Eulerian if and only if it is even and the (underlying multigraph of the directed) graph is connected.*

Proof. The statement follows from Observation 1 since any connected collection of disjoint (directed) cycles may be combined into a (directed) closed trail. □

A connected graph is Eulerian if one can walk through all its edges exactly once and return back to the origin. Theorem 1 provides a characterisation of Eulerian graphs, and its proof also gives an efficient algorithm for finding an Euler tour. It turns out that the problem changes drastically if we want to walk through *each vertex* exactly once.

In the *travelling salesman problem (TSP)*, a salesman is to make a tour of n cities, at the end of which he has to return to the city he starts from. The cost of the journey between any two cities is known. The TSP asks for (the cost of) a least expensive tour. This basic discrete optimisation problem belongs to the class of NP*-complete problems*, where the existence of an efficient algorithm is considered unlikely. If the cost of the journey between a pair of cities is either 1 or $+\infty$, we get a problem of finding a cycle in a graph, that goes through all the vertices. Such a cycle is *Hamiltonian*; deciding whether a graph has a Hamiltonian cycle is also an NP-complete problem.

2 Cycle space and cut space

Let $G = (V, E)$ be a graph. The *incidence matrix* I_G of G is the $(V \times E)$-matrix defined by $(I_G)_{ve} = 1$ if $v \in e$ and $(I_G)_{ve} = 0$ otherwise. If $A \subseteq E$, the *incidence vector* $i(A)$ of A is the vector indexed by E, where $[i(A)]_e = 1$ if $e \in A$ and $[i(A)]_e = 0$ otherwise. The *symmetric difference* of two sets A and B is

$$A \triangle B := \{x \in (A \setminus B) \cup (B \setminus A)\}\,.$$

Note that taking the symmetric difference of two subsets of edges amounts to summing their incidence vectors modulo 2. We usually do not make the distinction between a set and its incidence vector.

Let $\mathcal{K}$ be the set of all even sets of edges (and also the set of the incidence vectors of the even sets of edges). The next observation follows from the fact that $z \in \mathcal{K}$ if and only if each degree of (V, z) is zero modulo 2.

Observation 2. $\mathcal{K} = \{z \in \{0,1\}^E : I_G z = 0\}$*, where the equality is modulo* 2*.*

This means that $\mathcal{K}$ together with the operation of symmetric difference (or sum modulo 2 on the incidence vectors) is a vector space over the 2-element field $\mathbf{F}_2$. It is the *cycle space* of the graph G. Since each even set is a disjoint union of cycles, the set of the cycles of G generates the cycle space. Let us state this as a proposition.

Proposition 2. *Let $G = (V, E)$ be a graph and $F \subseteq E$. The following assertions are equivalent.*

(i) F is even;

(ii) F is a disjoint union of (edge sets of) cycles of G; and

(iii) $I_G \cdot i(F) = 0$.

Proof. We already observed that (i)⇔(iii). Further, observe that (ii)→(i). Finally, (i)→(ii) can be proved by induction on the size of F. Indeed, if $F \neq \emptyset$, then (V, F) contains a cycle. Remove its edges from F to obtain a set $F' \subset F$. Then (V, F') is even. So, if $F' = \emptyset$ then (ii) holds trivially, and otherwise applying the induction hypothesis to F' yields (ii). □

Let us generalise this construction.

Definition 1. A set $E' \subseteq E$ is an *edge-cut* if there is a partition of V into two sets V_1 and V_2 so that $E' = \{uv \in E : u \in V_1 \text{ and } v \in V_2\}$.

Example 1. Each non-empty edge-cut separates G, and each separating set of edges contains a non-empty edge-cut. The empty set is the edge-cut corresponding to the trivial bipartition $(V, \emptyset)$ of V.

If D is an arbitrary orientation of G we let I_D be the $(V \times E)$-matrix satisfying $(I_D)_{ve} = 1$ if e starts at v, $(I_D)_{ve} = -1$ if e ends at v, and $(I_D)_{ve} = 0$ otherwise. Let $\mathbf{F}$ be an arbitrary field and let A be a matrix over $\mathbf{F}$. The *kernel* of A is the set $\mathscr{K} := \{x \in \mathbf{F}^E : Ax = 0\}$ and the *image* of A is the set $\mathscr{I} := \{xA : x \in \mathbf{F}^V\}$. The kernel and the image of a matrix are orthogonal complements of each other.
The kernel of I_D over $\mathbf{F}$ is the *cycle space* of G over $\mathbf{F}$ and the image of I_D is the *cut space* of G over $\mathbf{F}$. Hence the cycle space and the cut space are orthogonal complements. We note next that the cycle spaces corresponding to different orientations of G are isomorphic, and the same holds for the cut spaces. To this end, we introduce fundamental cycles and cuts.

Let $G = (V, E)$ be a graph, and $T \subseteq E$ a largest acyclic set of edges of G. Hence, $|T| = |V| - k$, where k is the number of connected components of G. Further, for each edge $e \in T$, the subgraph spanned by $T - e$ has exactly $k + 1$ connected components. Thus, the edges of G between those components for a cut, D^e. These cuts D^e for $e \in T$ are the *fundamental cuts* of G (with respect to T). There are exactly $|V| - k$ fundamental cuts with respect to every maximum acyclic set of edges.

On the other hand, for each $e \in E \setminus T$, the set $T \cup \{e\}$ contains exactly one cycle, C^e. These cycles are the *fundamental cycles* of G (with respect to T). There are exactly $|E| - |V| + k$ fundamental cycles with respect to each maximum acyclic set of edges.

As we note next, fundamental cycles and fundamental cuts are basis of the cycle space and the cut space, respectively.

Observation 3. *Let $G = (V, E)$ be a graph and $D = (V, A)$ an orientation of G. For every maximum acyclic set of edges T, the fundamental cycles and the fundamental cuts (with respect to T) generate the cycle space and the cut space, respectively. In particular,*

$$\dim(\mathscr{I}) = |V| - k$$

and

$$\dim(\mathscr{K}) = |E| - |V| + k\,,$$

where k is the number of the connected components of G. Moreover, the cycle spaces over $\mathbf{F}$ of all the possible orientations of G are isomorphic (and hence so are the cut spaces).

Proof. Let $T \subseteq E$ be a maximum acyclic set of edges of G. Let $\mathscr{B}$ and $\mathscr{B}'$ be the sets of fundamental cycles and fundamental cuts with respect to T, respectively. The vectors in $\mathscr{B}$ are independent over $\mathbf{F}$, since for each $e \in E \setminus T$ the coordinate corresponding to e is 0 in all the vectors of $\mathscr{B}$ except C^e. Consequently,

$$\dim(\mathscr{K}) \geq |\mathscr{B}| = |E| - |V| + k\,.$$

Similarly, the vectors in $\mathscr{B}'$ are independent over $\mathbf{F}$, since each vector S^e is zero on the coordinates of all the edges in T except e. Consequently,

$$\dim(\mathscr{I}) \geq |\mathscr{B}'| = |T| = |V| - k\,.$$

Now, by the definition of the cut space and the cycle,

$$\dim(\mathscr{K}) + \dim(\mathscr{I}) = \dim(\mathbf{F}^E) = |E|\,.$$

Therefore, $\dim(\mathscr{K}) = |E| - |V| + k$ and $\dim(\mathscr{I}) = |V| - k$. Hence, the sets $\mathscr{B}$ and $\mathscr{B}'$ are basis of $\mathscr{K}$ and $\mathscr{I}$, respectively. □

As a result, note that the induced cycles of a graph generate its cycle space.

Let us make another remark on the cycle and cut spaces. Fix a field $\mathbf{F}$. Let $G = (V, E)$ be a graph and $D = (V, A)$ an orientation of G. A *circulation* (of G over $\mathbf{F}$) is a function $f := E \to \mathbf{F}$ such that for every vertex $v \in V$,

$$\sum_{e=u\to v\in A} f(e) = \sum_{e=v\to u} f(e)\,.$$

The cycle space of a graph G over $\mathbf{F}$ is the set of all the circulations over $\mathbf{F}$ of an orientation $D = (V, A)$ of G. If $\mathbf{F} = \mathbf{F}_2$, then the circulations coincide with the even sets of arcs of G. We can make an analogous observation for the cut space. Let p be a *potential* on G, i.e. a mapping from V to $\mathbf{F}$. Given an orientation $D = (V, A)$ of G, the corresponding *potential difference* is the function $p' : E \to \mathbf{F}$ defined by $p'(e = uv) := p(v) - p(u)$ whenever $u \to v \in A$. If $\mathbf{F} = \mathbf{F}_2$ then the potential differences coincide with the incidence vectors of the edge-cuts.

Max-Cut and Min-Cut problems belong to the basic hard problems of computer science. Given a graph $G = (V, E)$ with a (rational) weight $w(e)$ assigned to each edge $e \in E$, the *Max-Cut problem* asks for the maximum value of $\sum_{e\in E'} w(e)$ over all edge-cuts E' of G, while the *Min-Cut problem* asks for the minimum of the same function. The Max-Cut problem is NP-complete for non-negative edge-weights and hence both Max-Cut and Min-Cut problems are hard for general rational edge-weights. The Min-Cut problem is efficiently (polynomially) solvable for non-negative edge-weights. This has been a fundamental result of computer science, known as the Max-flow/Min-cut algorithm. Still, there are some special important classes of graphs where the general Max-Cut problem is efficiently solvable. One such class is the class of the planar graphs.

3 Planar graphs and graphs on surfaces

Let us describe some results on *planar representations*, i.e. drawings (*embeddings*) of graphs in the plane so that the vertices are represented by distinct points, each edge is represented by a continuous curve between the representations of the end-vertices of the edge, and the interior of each edge-representation is disjoint with the rest of the graph representation (i.e., in particular the edges do not cross each other). The graphs that can be represented (*embedded*) in the plane $\mathbf{R}^2$ in such a way are *planar*. A *plane graph*, or *topological plane graph* is a planar graph along with a given embedding in $\mathbf{R}^2$. We deal with the planar embeddings in an intuitive way. More background on embeddings is found in the book by Mohar and Thomassen [1]. In particular, consult it for all omitted proofs of this section.

Let us start by stating the very intuitive (yet definitely non-trivial) Jordan Curve Theorem. A *simple closed curve* of $\mathbf{R}^2$ is the image of a continous map from the sphere $\mathcal{S}^1$ into $\mathbf{R}^2$.

Theorem 3. *Any simple closed curve C divides the plane into exactly two connected components. Both of these regions have C as the boundary.*

A curve in the plane is a *polygonal arc* if it is the union of a finite number of straight line segments. The following lemma is very intuitive.

Lemma 4. *Every planar graph may be embedded into the plane so that all edges are represented by simple polygonal arcs.*

By the definition, each plane graph is a subset of the plane. A *face* of a plane graph is any connected component of its planar complement. Exactly one face of a plane graph G is unbounded, the *outer face* of G. Given a planar graph G we let n, m and f be its number of vertices, edges and faces, respectively. These numbers are interconnected by the following Euler's formula.

Theorem 5. *Let G be a connected plane graph. Then*

$$n - m + f = 2\,.$$

Proof. Fix the number $n \geq 1$ of vertices of G. Since G is connected, $m \geq n-1$. We proceed by induction on $m \geq n-1$. If $m = n-1$ then G is a tree. Hence, $f = 1$ and the formula is correct. Suppose that $m > n-1$ and the formula is true for all connected plane graphs with n vertices and $m' < m$ edges. Then, G contains a cycle, and so G contains an edge e such that $G - e$ is connected. Consequently, $G - e$ is a plane graph with n vertices, $m-1$ edges and $f-1$ faces. By the induction hypothesis, $n - (m-1) + (f-1) = 2$, which concludes the proof. □

Definition 2. Given a plane graph, a *facial cycle* is a set of edges that is a cycle and bounds a face.

Theorem 6. *Let G be a 2-connected plane graph. Then each face is bounded by a cycle of G and each edge belongs to the facial cycles of exactly two faces.*

Proof. We proceed by induction on the number of vertices of G, the result being true for cycles. Suppose that G is a 2-connected graph with $n > 3$ vertices. The graph G has an edge e so that $G - e$ is 2-connected, or G has a vertex v of degree 2. In the former case, applying the induction hypothesis to $G - e$ yields the result, while in the latter case the induction hypothesis is applied to the graph obtained from G by contracting an edge incident with v (and removing the parallel edge that may arise). □

What is the maximum number of edges of a planar graph with n vertices? Quite intuitively the maximum is achieved for a plane graph where each face is a triangle. A plane graph each face of which is a triangle is a *plane triangulation*. In such a graph, every face contains exactly three edges and each edge is in exactly two faces. Thus, $3f = 2m$. Now, applying Euler's formula yields that $2 = n - m + f = n - m + \frac{2}{3} \cdot m$, and thus $m = 3n - 6$. Hence, each planar graph with n vertices has at most $3n - 6$ edges. In particular, each planar graph has a vertex of degree at most 5.

Which graphs are not planar? The previous observation implies that K_5 is not planar. Using the Jordan Curve Theorem, one can show that $K_{3,3}$ is not planar either. Kuratowski's theorem states that these are in fact the only essential non-planar graphs.

An important concept is that of the *dual graph* G^* of a plane graph G. It turns out to be convenient to define G^* as an abstract (not topological) graph. But we need to allow multiple edges and loops, which is not included in the concept of the graph as a pair (V, E), where $E \subseteq \binom{V}{2}$. A standard way out is to define a graph as a triple (V, E, g) where V, E are sets and g is a function from E to $\binom{V}{2} \cup V$ which gives to each edge its endvertices. For instance, $e \in E$ is a loop if and only if $g(e) \in V$. Now we can define G^* as triple $(F(G), \{e^* : e \in E(G)\}, g)$ where $F(G)$ is the set of the faces of G and $g(e^*) = \{f \in F(G) : e \text{ belongs to the boundary of } f\}$.

It is important that the dual graph is defined with respect to an embedding. In fact, a planar graph may have several non-isomorphic dual graphs associated with it, corresponding to its different embeddings in the plane.

If G is a topological planar graph then G^* is planar. There is a natural way to properly draw G^* to the plane: represent each dual vertex $f \in F(G)$ as a point on face f, and represent each dual edge e^* by a curve between the points representing its endvertices, which crosses exactly once the representation of e in G and is disjoint from the rest of the representations of both G and G^*.

Recall that a set of edges is even if it induces even degree at each vertex. A subset E' of edges of a plane graph G is *dual even* if $\{e^* : e \in E'\}$ is an even set of edges of G^*.

Observation 4. *The dual even subsets of edges of G are exactly the edge-cuts of G.*

We can actually be more precise. We saw that cycles and edge-cuts are dual from an algebraic point of view. We now show that they are also dual from a geometric point of view.

Proposition 7. *Let $G = (V, E)$ be a connected plane multigraph. A set $F \subseteq E$ is a cycle in G if and only if F^* is a minimal edge-cut in G^*.*

Proof. By the definition of G^*, two vertices f_1^* and f_2^* of G^* are in the same component of $G^* - F^*$ if and only if the corresponding faces f_1 and f_2 of G are in the same region of $\mathbf{R}^2 \setminus F$. Indeed, a $f_1^* f_2^*$-path in G^* is an arc from f_1 to f_2 in $\mathbf{R}^2 \setminus F$, and conversely if there is an arc in $\mathbf{R}^2 \setminus F$ from f_1 to f_2, then we can choose one avoiding V and hence obtain a $f_1^* f_2^*$-path in G^*.

As a result, if F is a cycle in G, then we deduce from the Jordan Curve Theorem that $G^* - F^*$ has exactly two components, and hence F^* is a minimal edge-cut of G^*.

Conversely, suppose that F^* is a minimal edge-cut of G^*. Then $\mathbf{R}^2 \setminus F$ is not connected, therefore F^* cannot span a forest in G^* (since a forest has only one face). So F^* contains a cycle, and by the previous implication it cannot contain any further edge since F^* is minimal. □

Proposition 8. *Fix $k \in \{2, 3\}$ Let G be a plane graph and G^* its dual graph. If G is k-connected then so is G^*.*

Proof. Exercise! □

Now we are ready to describe surfaces. A *surface* is a connected compact Hausdorff topological space S which is locally homeomorphic to an open disc in the plane, i.e., each point of S has an open neighborhood homeomorphic to the open unit disc in $\mathbf{R}^2$. The next theorem expresses that we can get all surfaces by glueing together triangles.

Theorem 9. *Every surface has a finite triangulation of dimension* 2*.*

Let us consider two disjoint triangles T_1, T_2 with all sides equal, in a 2-simplex F of a triangulation of a surface S. We can make a new surface S' from S by deleting from F the interiors of T_1, T_2 and identifying T_1 with T_2 such that the clockwise orientations (in $F \subseteq \mathbf{R}^2$) around T_1 and T_2 disagree. The surface S' is obtained from S by *adding a handle*. There is another possibility of identifying T_1 with T_2; The resulting surface S'' is obtained from S by *adding a twisted handle*. Finally let T be a quadrangle (with equilateral sides) in F. We let S''' be the surface obtained from S by deleting the interior of T and identifying diametrically opposite points of the quadrangle. The surface S''' is obtained from S by *adding a crosscap*.

Let us consider now all surfaces obtained from the sphere S_0 (which we can think of here as a tetrahedron) by adding handles, twisted handles and crosscaps. If we add h handles to S_0, we obtain S_h, the *orientable surface of genus h*. If we add h crosscaps to S_0 we obtain N_h, the nonorientable surface of genus h. Surface S_1 is the *torus* (the doughnut surface), N_1 is the *projective plane*, and N_2 is the *Klein bottle*. The Klein bottle cannot be realised as a subset of $\mathbf{R}^3$.

The location and the order of adding handles and crosscaps is not important: the resulting surface is always the same, up to homeomorphism. Adding a twisted handle amounts to the same, up to homeomorphism, as adding two crosscaps. Moreover, if

we have already added a crosscap, then adding a handle amounts to the same, up to homeomorphism, as adding a twisted handle. In particular, if S is the surface obtained from the sphere by adding h handles, t twisted handles and c crosscaps then $S = S_h$ provided $t = c = 0$ and $S = N_{2h+2t+c}$ otherwise.
Now we are ready to state the Classification Theorem for surfaces.

Theorem 10. *Every surface is homeomorphic to precisely one of the surfaces S_h or N_k.*

Next we extend the concept of a triangulation of dimension 2 to embeddings of graphs. Let X be a topological space. Analogously as in the Euclidean space, *a curve* in X is the image of a continuous function $f : [0, 1] \to X$. The curve is *simple* if f is one-to-one, and it *connects* its endpoints $f(0)$ and $f(1)$. A curve is *closed* if $f(0) = f(1)$. A topological space is *(arcwise) connected* if any two elements are connected by an arc in X. A set $C \subseteq X$ *separates* X if $X - C$ is not connected. A *face* of $C \subseteq X$ is a maximal connected component of $X - C$.
A graph G is *embedded* in a topological space X if the vertices of G are distinct elements of X and every edge of G is a simple arc connecting its two endvertices in X and such that its interior is disjoint from other edges or vertices. Every graph has an embedding in $\mathbf{R}^3$. A graph embedded in a topological space X is also a *topological graph*. If G is a topological graph then $F(G)$ is the set of its faces.
The notion generalising triangulations is that of maps.

Definition 3. A *map* is a topological graph embedded on an orientable surface so that each face is homeomorphic to an open disc in the plane.

The next observation is Euler's formula.

Lemma 11. *Let S_h be an orientable surface of genus h and let G be a map in S_h with n vertices, e edges and f faces. Then $n - e + f = 2 - 2h$.*

The genus of a map is usually defined as the genus of the surface where the map exists. Lemma 11 allows us to define the *Euler characteristic* $\chi(S_h)$ of the surface S_h in analogy to simplicial complexes by setting $\chi(S_h) := 2 - 2h$.

What is a map? The definition of a map is presented above, but the truth is that it is a notion that confuses people. Edmonds realised that maps may be defined purely combinatorially.

Theorem 12. *There is a natural bijection between maps and the connected graphs decorated with fixed cyclic orderings of the incident edges of each vertex.*

Physicists sometimes prefere *fatgraphs* to maps. This term corresponds to a helpful graphic representation of a graph (not necessarily connected), in which the vertices are made into discs (islands) and connected by fattened edges (bridges) prescribed by the cyclic orderings of the incident edges of each vertex. This defines a two-dimensional orientable surface with boundary. Let F ve a fatgraph, and also the corresponding surface with bounded by F. Each component of the boundary of F is a *face* of F. Each face is an embedded circle. We let $G(F)$ be the underlying graph of F. Define $e(F), n(F), f(F), c(F)$ and $g(F)$ to be the number of edges, vertices, faces, connected components, and the genus of F, respectively. Euler's formula can be rewritten for for fatgraphs.

Lemma 13. $n(F) - e(F) + f(F) = 2(c(F) - g(F))$.

An important concept is that of the *dual graph* G^* of a (general) topological graph G. It may be defined for general topological graphs in exactly the same way as for the topological planar graphs.

References

[1] B. Mohar and C. Thomassen. *Graphs on surfaces.* John Hopkins Univer-sity Press, Baltimore, 2001.