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Lecture 2

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1 Ising problem, cuts and even sets

A classical example of a statistical physics model is the *Ising model*. It was designed by Lenz around 1920 to explain ferromagnetism and was named after his student Ising who contributed to its theory. The idea is that iron atoms are situated at vertices of a graph $G = (V, E)$ and behave like small magnets that can be oriented upwards (have *spin* $+1$) or downwards (have *spin* -1). Physically, two magnets that are close to one another need less energy to be oriented in the same way than in the opposite way. This leads to the following simplified model.

We assume that each edge uv of G has an assigned weight (a *coupling constant*) $w(uv)$. A configuration of the system is an assignment of the *spin* $\sigma_v \in \{+1, -1\}$ to each vertex v. This describes the two possible spin orientations the vertex can take. As is customary in statistical physics, the configurations of the Ising model are its *states*.

The *energy function* (or *Hamiltonian*) of the model is defined as

$$z(\sigma) := -\sum_{uv\in E} w(uv)\sigma_u\sigma_v\,.$$

Hence,

$$z(\sigma) = \sum_{uv\in C} w(uv) - \sum_{u,v\in E\setminus C} w(uv) = 2w(C) - W\,,$$

where C is the set of edges connecting the pairs of vertices of different spins, and $W := \sum_{u,v\in E} w(uv)$ is the sum of all edge weights in the graph. If we find the value of a maximum cut then we have found the maximum energy of the model. Similarly, the Min-Cut problem corresponds to finding the minimum energy (*groundstate*). The *partition function* is

$$Z(G, \beta) := \sum_{\sigma} e^{-\beta z(\sigma)}\,.$$

In the next section we describe an efficient algorithm to determine the groundstate energy of the Ising model for any planar graph. In fact the whole partition function may be determined efficiently for planar graphs, and the principal ingredient is the concept of enumeration dualities of Section 3, via the theorems of van der Waerden and MacWilliams.

2 Max-Cut for planar graphs

We saw in the first lecture that the dual even sets of edges of a planar graph G are precisely the edge-cuts of G (recall that a set of edges of G is dual even if it spans an even subgraph of the dual G^* of G). Thus, for planar graphs, the Max-Cut problems reduces to the following problem.

Maximum even subset problem. Given a graph $G = (V, E)$ with rational weights on the edges, find the maximum value of $\sum_{e \in E'} w(e)$ over all even subsets $E' \subseteq E$.

The following theorem thus means that the Max-Cut problem is efficiently solvable for planar graphs.

Theorem 1. *The maximum even subset problem is efficiently solvable for general graphs.*

The theorem is proved by a reduction to the *matching problem*. An efficient algorithm to find a *maximum matching* of a graph has been found by Edmonds (see, e.g., the book by Lovász and Plummer [1] for a description of it). Edmonds also found an efficient algorithm to solve the following generalisation of the matching problem.
Weighted perfect matching problem. Given a graph $G = (V, E)$ with rational weights

on the edges, the weighted perfect matching problem asks for the maximum value of $\sum_{e \in E'} w(e)$ over all perfect matchings E' of G.
This algorithm together with the following reduction of Fisher proves Theorem 1.

Theorem 2. *Given a graph $G = (V, E)$ with a weight function w on E, it is possible to construct graph $G' = (V', E')$ and a weight function w' on E' so that there is a natural weight preserving bijection between the set of the even sets of G and the set of the perfect matchings of G'.*

Proof. The graph G' may be constructed from G by a local transformation at each vertex, ... We let $w'(e) = w(e)$ for all edges e of G, and $w'(e) = 0$ for each new edge. □

In statistical physics, a perfect matching is known as a *dimer arrangement* or a *dimer configuration*. The *dimer problem* is the problem to enumerate all perfect matchings, i.e., to find the *partition function*

$$\mathscr{P}(G, x) := \sum_{E' \text{ perfect matching}} x^{\sum_{e \in E'} w(e)}.$$

In discrete mathematics, partition functions are called *generating functions*. The bijection of Theorem 2 does more than just a reduction of the maximum even set to the maximum perfect matching. The whole information about the even sets is

transformed to the information about the perfect matchings. This may be expressed by the equality of the generating functions.
Let G be a graph and let w be a weight function on E. For $E' \subseteq E$ let $w(E') := \sum_{e \in E'} w(e)$. We define the generating functions $\mathscr{C}(G,x), \mathscr{E}(G,x), \mathscr{P}(G,x)$ of the edge-cuts, the even sets and the perfect matchings, respectively, as follows.

$$\mathscr{C}(G,x) := \sum_{E' \text{ edge-cut}} x^{w(E')},$$

$$\mathscr{E}(G,x) := \sum_{E' \text{ even set}} x^{w(E')},$$

$$\mathscr{P}(G,x) := \sum_{E' \text{ perfect matching}} x^{w(E')}.$$

The transformation of Theorem 2 shows that $\mathscr{E}(G,x) = \mathscr{P}(G',x)$.

3 Enumeration duality

The Ising partition function for a graph G may be expressed in terms of the generating function of the even sets of the same graph G. This is a theorem of van der Waerden. We use the following standard notation: $\sinh(x) = (e^x - e^{-x})/2$, $\cosh(x) = (e^x + e^{-x})/2$ and $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$.

Theorem 3.

$$Z(G,\beta) = 2^{|V|} \left(\prod_{uv \in E} \cosh(\beta w(uv)) \right) \left(\sum_{\substack{E' \subseteq E \\ E' \text{ even}}} \prod_{uv \in E'} \tanh(\beta w(uv)) \right).$$

Proof. We have

$$\begin{aligned} Z(G,\beta) &= \sum_{\sigma} e^{\beta \sum_{uv} w(uv) \sigma_u \sigma_v} \\ &= \sum_{\sigma} \prod_{uv \in E} (\cosh(\beta w(uv)) + \sigma_u \sigma_v \sinh(\beta w(uv))) \\ &= \prod_{uv \in E} \cosh(\beta w(uv)) \sum_{\sigma} \prod_{uv \in E} (1 + \sigma_u \sigma_v \tanh(\beta w(uv))) \\ &= \prod_{uv \in E} \cosh(\beta w(uv)) \sum_{\sigma} \sum_{E' \subseteq E} \prod_{uv \in E'} \sigma_u \sigma_v \tanh(\beta w(uv)) \\ &= \prod_{uv \in E} \cosh(\beta w(uv)) \sum_{E' \subseteq E} (U(E') \prod_{uv \in E'} \tanh(\beta w(uv))), \end{aligned}$$

where

$$U(E') := \sum_{\sigma} \prod_{uv \in E'} \sigma_u \sigma_v \,.$$

The proof is complete when we notice that $U(E') = 2^{|V|}$ if E' is even and $U(E') = 0$ otherwise. Indeed,

$$U(E') = \sum_{\sigma} \prod_{u \in V} (\sigma_u)^{\deg_{E'}(u)} \,.$$

Consequently, if E' is even then $U(E') = 2^|V|$, while if $u \in V$ has odd degree in (V, E') then $U(E')$ is the number of states σ such that $\sigma_u = 1$ minus the number of states σ such that $\sigma_u = -1$, which is 0. □

It may (optically) help to shorten the right-hand-side of the formula by

$$\sum_{\substack{E' \subseteq E \\ E' \text{ even}}} \prod_{uv \in E'} \tanh(\beta w(uv))) = \mathscr{E}(G, x)|_{x^{w(uv)} := \tanh(\beta w(uv))} \,,$$

where $\mathscr{E}(G, x)$ is the generating function of even subsets introduced at the end of Section 2. Let us recall that we may write the partition function as

$$Z(G, \beta) = \sum_{\sigma} e^{\beta \sum_{uv} w(uv) \sigma_u \sigma_v} = K \sum_{\sigma} e^{-2\beta \sum_{uv : \sigma(u) \neq \sigma(v)} w(uv)} \,,$$

where $K := \sum_{\sigma} e^{\beta \sum_{uv} w(uv)}$ is a constant.
Hence the partition function $Z(G, \beta)$ may be seen as the generating function of edge-cuts *with specified shores*. The theorem of van der Waerden expresses it in terms of the generating function $\mathscr{E}(G, x)$ of the even sets of edges.
We can also consider the honest *generating function of edge-cuts*

$$\mathscr{C}(G, x) := \sum_{\substack{E' \subseteq E \\ E' \text{ edge-cut}}} x^{w(E')} \,.$$

It turns out that $\mathscr{C}(G, x)$ may also be expressed in terms of $\mathscr{E}(G, x)$. This is a consequence of another theorem, of MacWilliams, which we explain now.

Let $V := \mathbf{F}^n$ be a finite vector space over a finite field $\mathbf{F}$. Each subspace C of V of dimension k is a *linear code of length n and dimension k*. If $\mathbf{F} = \mathbf{F}_2$ then C is a *binary (linear) code*. The elements of a linear code are *codewords*. The *weight* of a codeword is the number of its nonzero entries. The *weight distribution* of C is the sequence $A_0, A_1, \cdots, A_n$ where A_i equals the number of codewords of C of weight i for $i \in \{0, 1, \ldots, n\}$.
The *dual code* C^* of C is composed of all the n-tuples $(d_1, \cdots, d_n)$ of $\mathbf{F}^n$ satisfying

$$c_1 d_1 + \cdots + c_n d_n = 0$$

in $\mathbf{F}$, for all codewords $(c_1, \cdots, c_n) \in C$. Hence, C^* is a code of length n and dimension $n-k$. The *weight enumerator* of C is the polynomial

$$A_C(t) := \sum_{i=0}^{n} A_i t^i \,.$$

MacWilliams' theorem for $\mathbf{F} = \mathbf{F}_2$ reads as follows.

Theorem 4.

$$A_{C^*}(x) = \frac{1}{|C|}(1+x)^n A_C\left(\frac{1-x}{1+x}\right) \,.$$

The proof is by a series of lemmas. We start by defining the *extended generating function of a code.*

Definition 1. Let $C \subseteq \{0,1\}^n$ be a binary linear code and $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$ be variables. The *extended generating function* of C is

$$W_C(x,y) := \sum_{b \in C} W_b(x,y) \,,$$

where for $b = (b_1, \cdots, b_n)$, we set $W_b(x,y) := \prod_{i=1}^{n} W_{b_i}(x_i, y_i)$ with $W_{b_i}(x_i, y_i) := x_i$ if $b_i = 0$ and y_i if $b_i = 1$.

Lemma 5. *Let $b \in \{0,1\}^n$. Then*

$$W_b(x+y, x-y) = \sum_{c \in \{0,1\}^n} (-1)^{bc} W_c(x,y) \,.$$

Proof. We note that

$$W_b(x+y, x-y) = \prod_{i=1}^{n} W_{b_i}(x_i + y_i, x_i - y_i) = \prod_{i=1}^{n} (x_i + (-1)^{b_i} y_i) \,.$$

Expanding the right hand side we obtain a sum of 2^n terms of the form $\pm z_1 \cdots z_n$ where $z_i = x_i$ or $z_i = y_i$ and the sign is negative if and only if there is an odd number of indices i where $z_i = y_i$ and $b_i = 1$. Letting c index this sum of 2^n terms we obtain

$$\prod_{i=1}^{n} (x_i + (-1)^{b_i} y_i) = \sum_{c \in \{0,1\}^n} (-1)^{bc} \prod_{i=1}^{n} W_{c_i}(x_i, y_i) \,.$$

□

Lemma 6. *If $c \notin C^*$ then the sets $A_i = \{b \in C; cb = i\}$ have the same cardinality for $i \in \{0,1\}$.*

Proof. We first note that both sets A_i are non-empty: $0 \in A_0$ and since $c \notin C^*$, there is $b \in C$ such that $bc = 1$. Let $b \in A_1$. Then $|b + A_0| = |A_0|$ and $b + A_0 \subseteq A_1$. Hence $|A_0| \leq |A_1|$. Analogously, $|-b + A_1| = |A_1|$ and $-b + A_1 \subseteq A_0$. Hence $|A_1| \leq |A_0|$. □

Lemma 7.

$$W_{C^*}(x, y) = \frac{1}{|C|} W_C(x + y, x - y)\,.$$

Proof.

$$\begin{aligned} W_C(x + y, x - y) &= \sum_{b \in C} \sum_{c \in \{0,1\}^n} (-1)^{bc} W_c(x, y) \\ &= \sum_{b \in C} \sum_{c \in C^*} W_c(x, y) + \sum_{b \in C} \sum_{c \notin C^*} (-1)^{bc} W_c(x, y) \\ &= |C| W_{C^*}(x, y) \end{aligned}$$

by Lemma 6. □

Lemma 8.

$$x^n A_C(y/x) = W_C(x, \ldots, x, y, \ldots, y)\,.$$

Proof. We observe that

$$W_C(x, \ldots, x, y, \ldots, y) = x^n \sum_{b \in C} (y/x)^{w(b)}\,,$$

where $w(b)$ is the weight of b. □

Proof of Theorem 4: By Lemma 8,

$$A_{C^*}(y) = W_{C^*}(1, \cdots, 1, y, \cdots, y)\,.$$

Next we apply Lemma 7 and Lemma 8 again. □ We saw that the set of the edge-cuts and the set of the even sets of edges form dual binary codes, hence MacWilliams' theorem applies here. Finally we note that a version of MacWilliams' theorem is true more generally, for instance for linear codes over finite fields $\mathbf{F}_q$.

References

[1] L. Lovász and M. D. Plummer. *Matching theory*, volume 121 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986. Annals of Discrete Mathematics, 29.