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Lecture 4	
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1 The permanent of a matrix

1.1 Minc's conjecture

The set of permutations of $\{1,\ldots,n\}$ is $\mathscr{S}_n$. Let $A=(a_{i,j})_{1\le i,j\le n}$ be a square matrix with real non-negative entries. The *permanent* of the matrix A is

$$\operatorname{perm}(A) := \sum_{\sigma\in\mathscr{S}_n}\prod_{i=1}^{n} a_{i,\sigma(i)}\,.$$

In 1973, Brègman [4] proved Mínc's conjecture [18].

Theorem 1 (Brègman, 1973)**.** *Let $A=(a_{i,j})_{1\le i,j\le n}\in\{0,1\}^{n\times n}$. Set $r_i := \sum_{j=1}^{n} a_{i,j}$. Then,*

$$\operatorname{perm}(A)\le\prod_{i=1}^{n}(r_i!)^{1/r_i}\,.$$

Further, if $r_i>0$ for every $i\in\{1,2,\ldots,n\}$, then there is equality if and only if up to permutations of rows and columns, A is a block-diagonal matrix, each block being a square matrix with all entries equal to 1.

Several proofs of this result are known, the original being combinatorial. In 1978, Schrijver [22] found a neat and short proof. A probabilistic description of this proof is presented in the book of Alon and Spencer [3, Chapter 2]. The one we will see in Lecture 5 uses the concept of entropy, and was found by Radhakrishnan [20] in the late nineties. It is a nice illustration of the use of entropy to count combinatorial objects.

1.2 The van der Waerden conjecture

A square matrix $M=(m_{ij})_{1\le i,j\le n}$ of non-negative real numbers is *doubly stochastic* if the sum of the entries of every line is equal to 1, and the same holds for the sum of the entries of each column. In other words,

$$\forall(i,j)\in\{1,2,\ldots,n\}^2,\quad \sum_{k=1}^{n} m_{ik}=\sum_{k=1}^{n} m_{kj}=1\,.$$

In 1926, van der Waerden conjectured that the permanent of every doubly stochastic matrix M of size $n \times n$ is at least $\frac{n!}{n^n}$, with equality if and only if $m_{ij} = \frac{1}{n}$ for all pairs $(i, j) \in \{1, 2, \ldots, n\}^2$. This was proved more than fifty years later by Egorychev [7, 8, 9] and, independently, Falikman [10]. See also the work of Gyires [12, 13].

Theorem 2 (Egorychev, Falikman, 1979-1980)**.** *If $M = (m_{ij})_{1 \le i,j \le n}$ is a doubly stochastic matrix then*

$$\mathrm{perm}(M) \ge \frac{n!}{n^n}\,,$$

with equality if and only if $m_{ij} = \frac{1}{n}$ for all pairs $(i, j) \in \{1, 2, \ldots, n\}^2$.

Forthwith some applications of those theorems to obtain bounds on the number of perfect matchings of certain graphs.

2 The number of perfect matchings

Let G be a graph. A *matching* of G is a set M of edges of G such that no two edges in M are adjacent in G. A matching M is *perfect* if every vertex of G is incident to an edge of M. We let $\mathrm{pm}(G)$ be the number of perfect matchings of G.

The permanent of a 0-1-matrix can be interpreted as the number of perfect matchings in a bipartite graph. More precisely, given such a matrix $A = (a_{ij})_{1 \le i,j \le n}$, we can define a bipartite graph G with two parts $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$, and there is an edge between $u_i \in U$ and $v_j \in V$ if and only if $a_{ij} = 1$. It directly follows from the definition of the permanent that

$$\mathrm{perm}(A) = \mathrm{pm}(G)\,.$$

Conversely, the number of perfect matchings of a bipartite graph is the permanent of its *incidence matrix*, i.e. if U and V are the two color classes, the matrix is $(a_{uv})_{(u,v) \in U \times V}$ with $a_{uv} = 1$ if uv is an edge, and 0 otherwise.

We focus on bounding the number of perfect matchings in some classes of graphs. It is an old question, the study of which began with the class of regular bipartite graphs. The first non-trivial lower bound on the number of perfect matchings in 3-regular bridgeless bipartite graphs was obtained in 1969 by Sinkhorn [25], who proved a bound of $\frac{n}{2}$. He thereby established a conjecture of Marshall. The same year, Minc [19] increased this lower bound by 2 and one year after, Hartfiel [14] obtained $\frac{n}{2} + 3$. Next, Hartfiel and Crosby [15] improved the bound to $\frac{3}{2}n - 3$. The first exponential bound was obtained in 1979 by Voorhoeve [26], who proved $6 \cdot \left(\frac{4}{3}\right)^{n/2-3}$. This was generalised to all regular bipartite graphs in 1998 by Schrijver [23], who thereby proved a conjecture of himself and Valiant [24]. His argument is highly involved, and the obtained bound is a bit weaker than Voorhoeve's when applied to 3-regular graphs. As a particular case of a different and more general approach (using hyperbolic polynomials), Gurvits [11] managed to slightly improve the bound (in particular, Gurvits' bound matches Voorhoeve's when applied to 3-regular graphs), as well as simplify the proof. His main result unifies (and generalises) the conjecture of Schrijver and Valiant with that of van der Waerden on the permanent of doubly stochastic matrices.

The problem of lower bounding the number of perfect matchings is also related to the following conjecture of Lovász and Plummer (see the book by Lovász and Plummer [17, Conjecture 8.1.8]).

Conjecture 1 (Lovász and Plummer, mid-1970s)*.* The number of perfect matching of a 3-regular bridgeless graph on n vertices is at least $2^{\Omega(n)}$.

Edmonds, Lovász, and Pulleyblank [6] proved that the dimension of the perfect matching polytope of a cubic bridgeless graph with n vertices is at least $n/4+1$. Since the vertices of the polytope correspond to distinct perfect matchings, it follows that any 3-regular bridgeless graph on n vertices has at least $\frac{n}{4}+2$ perfect matchings. A new lower bound of $\frac{n}{2}+2$ has been obtained [16], except for 17 exceptional graphs (one having exactly $\frac{n}{2}$ perfect matchings, the others $\frac{n}{2}+1$). Very recently, Esperet, Kardoš and Král' obtained the first super-linear bound, of order approximately $n \log n$.

In addition, Chudnovsky and Seymour [5] proved that Lovász and Plummer's conjecture is true for planar graphs. We will see in a future lecture the first part of their proof, which deals with cyclically 4-connected cubic planar graphs. Moreover, we will also prove the conjecture for the special class of fullerene graphs.

2.1 Graphs with a given degree sequence

We present in this section a tight upper bound on the number of perfect matchings in graphs with a given degree sequence. Its derivation as a simple consequence of Brègman's theorem is due to Alon and Friedland [1].

We set $(0!)^{1/0} := 0$. Before stating and proving the result, let us give some terminology. A subgraph H of a graph $G = (V, E)$ is *spanning* if every vertex of G has degree at least 1 in H. If $k \geq 3$, a *k-cycle* is a cycle of length k. Further, an edge uv is considered as a 2-cycle (namely the 2-cycle uvu).

Theorem 3 (Alon and Friedland, 2008)**.** *Let G be a graph with degree sequence $d_1, d_2, \ldots, d_n$. The number of perfect matchings of G is at most*

$$\prod_{i=1}^{n} (d_i!)^{1/(2d_i)} .$$

If G has no isolated vertices, then equality holds if and only if G is a disjoint union of complete balanced bipartite graphs.

Proof. Recall that $\mathrm{pm}(G)$ is the number of perfect matchings of G. Thus, $\mathrm{pm}(G)^2$ is the number of ordered pairs of perfect matchings of G. The union of two perfect matchings induces a subgraph of G the components of which are either even cycles or single edges. For convenience, we view single-edge components as 2-cycles. In other words, $\mathrm{pm}(G)^2$ is the number of spanning 2-regular subgraphs consisting of even cycles (where 2-cycles as defined above are allowed), each being counted 2^s times, where s is the number of k-cycles with k an even number greater than 2.

Let $A = (a_{uv})_{(u,v)\in V(G)^2}$ be the *adjacency matrix* of G, i.e. $a_{uv} = 1$ if uv is an edge of G, and 0 otherwise. The permanent of A counts the number of spanning 2-regular subgraphs, each being counted 2^s times, where s is the number of cycles of size greater than 2.

Consequently,

$$\mathrm{pm}(G) = \sqrt{\mathrm{pm}(G)^2} \le \sqrt{\mathrm{perm}(A)} \le \prod_{i=1}^{n} (d_i!)^{1/(2d_i)}\,,$$

the last inequality following from Brègman's theorem.

We now turn to the equality case. Suppose that the graph G has no vertex of degree 0 (i.e. no isolated vertex). The bound is surely attained if G is a disjoint union of complete balanced bipartite graphs. Conversely, if the equality is attained then the equality is also attained in Brègman's bound. Since G has no isolated vertex, no row of A sum to zero. Therefore, Brègman's theorem ensures that, up to permutations of rows and columns, A is a block-diagonal matrix in which each block is an all-1 square matrix. Since G has no loops, this means that G is a disjoint union of complete balanced bipartite graphs. □

3 Super (d,ε)-regular graphs

We prove a result of Alon, Rödl and Ruciński [2] on the number of perfect matchings in ε-regular graphs.

An *ε-regular* graph on $2n$ vertices is a bipartite graph composed of two color classes V_1 and V_2 of size n and such that for any two sets $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ of size at least εn each,

$$\left| \frac{e(U_1,U_2)}{|U_1|\cdot|U_2|} - \frac{e(V_1,V_2)}{|V_1|\cdot|V_2|} \right| < \varepsilon\,,$$

where $e(X,Y)$ is the number of edges between X and Y. The quantity $\frac{e(V_1,V_2)}{|V_1|\cdot|V_2|}$ is the *density* of the graph G. Further, G is *super (d,ε)-regular* if, in addition, its minimum degree δ and its maximum degree Δ satisfy

$$(d-\varepsilon)\cdot n \le \delta \le \Delta \le (d+\varepsilon)\cdot n\,.$$

Recall that a subgraph H of a graph G is spanning if every vertex of G is incident to an edge of H. A spanning subgraph which is k-regular is a *k-factor*.

Theorem 4 (Alon, Rödl and Ruciński, 1998)**.** *For every ε there exists n_0 such that for every $n > n_0$ and $d > 2\varepsilon$, the number $M(G)$ of perfect matchings of every super (d, ε)-regular graph G on $2n$ vertices satisfies*

$$(d - 2\varepsilon)^n n! \leq \mathrm{pm}(G) \leq (d + 2\varepsilon)^n n!\,.$$

Proof. Let G be a super (d, ε)-regular graph with bipartition (U, V). We first prove the upper bound. Let $A = (a_{uv})_{(u,v) \in U \times V}$ be the incidence matrix of G. By Brègman's theorem,

$$\mathrm{pm}(G) = \mathrm{perm}(A) \leq \prod_{u \in U} (\deg(u)!)^{1/\deg(u)} \leq \prod_{u \in U} (k!)^{1/k} = (k!)^{n/k}$$

with $k := \lfloor (d + \varepsilon) n \rfloor$. The announced upper bound follows by applying (twice!) Stirling's formula. (Note that we used neither the ε-regularity assumption nor the lower bound on the minimum degree.)

It remains to prove the lower bound. Let us note the following consequence of van den Waerden's bound. If H is a k-regular bipartite graph on $2n$ vertices, then

$$\mathrm{pm}(H) \geq \left(\frac{k}{n}\right)^n \cdot n!\,.$$

Indeed, let $B = (b_{ij})_{1 \leq i,j \leq n}$ be the incidence matrix of H. For every i, j, set $m_{ij} := \frac{1}{k} \cdot b_{ij}$. Since H is k-regular, the matrix M is doubly stochastic. Therefore, $\mathrm{perm}(M) \geq \frac{n!}{n^n}$. On the other hand, it follows from the definition of the permanent that $\mathrm{perm}(B) = k^n \cdot \mathrm{perm}(M)$, and thus

$$\mathrm{pm}(H) = \mathrm{perm}(B) \geq k^n \cdot \frac{n!}{n^n}\,.$$

As a result, it suffices to find a k-factor of G for $k := \lceil (d - 2\varepsilon) n \rceil$. We use the following condition for the existence of a k-factor. Its proof is given at the end of these notes.

The graph G has a k-factor if and only if any two sets $X \subseteq U$ and $Y \subseteq V$ satisfy

$$k \cdot |X| + k \cdot |Y| + e(X^c, Y^c) \geq kn\,, \tag{1}$$

where $X^c := U \setminus X$ and $Y^c := V \setminus Y$ (recall that $e(X^c, Y^c)$ is the number of edges of G with one endvertex in X^c and the other in Y^c).

Consider two sets $X \subseteq U$ and $Y \subseteq V$, and let us show that they satisfy the desired condition (in what follows, we forget the ceiling in the definition of k for convenience). Without loss of generality (by symmetry), we assume that $|X| \leq |Y|$. First, if $|X| + |Y| \geq n$, then (1) surely holds. Thus we assume now that $|X| + |Y| < n$. Next, suppose that $|Y^c| < \varepsilon \cdot n$. Then, $|Y| > (1 - \varepsilon)n$ and hence $|X| < \varepsilon n$. Consequently, every vertex of Y^c has at least $\delta - \varepsilon \cdot n \geq (d - 2\varepsilon)n = k$ neighbors in X^c. Thus,

$k \cdot |Y| + e(X^c, Y^c) \geq k \cdot |Y| + k \cdot |Y^c| = kn$ so that (1) holds. Finally, assume that $|Y^c| \geq \varepsilon \cdot n$. Thus, $|X^c| \geq \varepsilon \cdot n$. Since G is ε-regular,

$$\frac{e(X^c, Y^c)}{|X^c| \cdot |Y^c|} > \frac{e(U, V)}{|U| \cdot |V|} - \varepsilon\,,$$

and therefore

$$\begin{aligned} e(X^c, Y^c) >& \left(\frac{\delta \cdot n}{n^2} - \varepsilon\right) \cdot |X^c| \cdot |Y^c| \\ \geq& \frac{k}{n} \cdot |X^c| \cdot |Y^c|\,. \end{aligned}$$

It follows that

$$\begin{aligned} k \cdot |X| + k \cdot |Y| + e(X^c, Y^c) \geq& k(|X| + |Y|) + \frac{k}{n}(n - |X|)(n - |Y|) \\ \geq& k(|X| + |Y|) + kn - k \cdot |Y| - k \cdot |X| + \frac{k}{n} \cdot |X| \cdot |Y| \\ \geq& kn\,. \end{aligned}$$

Therefore, G has a k-factor, and thus at least $(d - 2\varepsilon)^n n!$ perfect matchings, as stated. □

Consider a random bipartite graph on two color classes of size n each, with edge probability d (i.e. for two sets $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$, the edge $u_i v_j$ is added with probability d independently of the other choices). The expected degree of such a graph is d, and the expected number of perfect matchings is $d^n n!$. Moreover, such a graph is ε-regular. Thus, Theorem 4 says that the number of perfect matching of a super (d, ε)-regular graph is close to the expected number of perfect matchings in a random bipartite graph. These ideas have been pushed further, and in particular Rödl and Ruciński [21] managed to obtain a new proof of the Blow-up lemma of Komlós, Sárközy and Szemerédi.

The condition for the existence of a k-factor that we used in the proof of Theorem 4 is a direct consequence of the max-flow/min-cut theorem, as we see next. A more general version of this theorem (where the degree of the spanning subgraph at each vertex is given by a function f) can be found in the book by Lovász and Plummer [17, Theorem 2.4.2].

Theorem 5. *Let G be a bipartite graph on $2n$ vertices with color classes U and V. Then, G has a k-factor if and only if $|U| = |V|$, and for every $X \subseteq U$ and every $Y \subseteq V$,*

$$k \cdot |X| + k \cdot |Y| + e(X^c, Y^c) \geq kn\,.$$

Proof. The fact that (1) holds if G has a k-factor can be directly checked. First, if G has a k-factor H then $|U| = |V| = n$ since the number of edges of H is equal to $k \cdot |U|$ and to $k \cdot |V|$. Let $X \subseteq U$ and $Y \subseteq V$. There are kn edges of H between U and V. Exactly $k|X^c|$ of these edges have an endvertex in X^c and the other in V. Among those edges from X^c to V, at most $k \cdot |Y|$ have an endvertex in Y. Thus,

$$kn = k \cdot |X| + k \cdot |X^c| \leq k \cdot |X| + k \cdot |Y| + e_H(X^c, Y^c)\,.$$

Consequently, (1) holds since $e_H(X^c, Y^c) \leq e_G(X^c, Y^c)$.

Conversely, assume that (1) holds and let us prove that G has a k-factor. Let D be the oriented graph obtained from G by orienting each edge of G from U to V, adding a source s joined to all the vertices of U, and a sink t joined to all the vertices of V (thus, s has out-degree $|U| = n$ while t has in-degree $|V| = n$). Each arc from U to V has capacity 1, while the other arcs all have capacity k. Observe that G has a k-factor if and only if D has an (integral) flow of value kn (since $|U| = |V| = n$). By the max-flow/min-cut theorem, D has such a flow if and only if every st-cut has capacity at least kn. Let S be an st-cut, that is S is a set of vertices of D such that $s \in S$ and $t \notin S$. Set $X := U \setminus S$ and $Y := V \cap S$. Then the capacity of S is $k \cdot |X| + e(X^c, Y^c) + k \cdot |Y|$, which is at least kn by hypothesis. This concludes the proof. $\square$

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