Lecture 9

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1 Introduction

As announced in the previous lecture, our goal is to prove the following theorem of Chudnovsky and Seymour [1]. It is the first step of their proof that Lovász and Plummer's conjecture holds for planar graphs.

Theorem 1 (Chudnovsky and Seymour, 2008)**.** *Every cyclically* 4*-connected planar cubic graph on* n *vertices has at least* $2^{n/92928}$ *perfect matchings.*

Before going into more details, let us recall the following equivalence proved by Tait [3] in 1880. The proof is left as an exercise; the reader can consult Diestel's book [2] or West's monograph [5] for more explanations on this.

Theorem 2. *Every planar graph is* 4*-colourable if and only if every planar bridgeless graph is* 3*-edge-colourable.*

The general idea of the proof is well-explained in the introduction of the original paper [1], to which we refer. In few words, the authors make the following remarks. First, the planarity brings two things: a source of triple of perfect matchings covering all edges (since the graphs we consider are 3-edge-colourable); second, the faces give us cycles that can be removed without drastically decreasing the connectivity of the graph. The proof of Theorem 1 goes as follows. Fix a cyclically 4-connected planar cubic graph G. A set of disjoint even cycles (which will be either even faces, or the union of two adjacent odd faces) of linear size is found. This set has exponentially many subsets. For every such subset X, and for each cycle C in X, we can delete the even edges of C and double the odd edges. The graph H_X we obtain is cubic and planar. So, if it has no cut-edge, then it is 3-edge-colourable and hence we obtain a triple of perfect matchings of the original graph G. A key observation (see the proof of Proposition 3) is that X can be reconstructed from this triple of perfect matchings. In particular, if we have k such subsets X, we would obtain $k^{1/3}$ distinct perfect matchings of G provided we can obtain for each subset X a planar cubic bridgeless graph H_X in the way described above. To this end, we will need to carefully choose the even and odd edges of each cycle C in X (which comes from the fact that the cycles in X may be too close to allow us an arbitrary choice).

We now introduce the notation used in the proof, using Chudnovsky and Seymour's terminology.

Given a graph G, we let $V(G)$ and $E(G)$ be its vertex-set and edge-set, respectively. For $X \subset V$, the set of edges of G with exactly one endvertex in X is $\delta(X)$. An *edge-cut* is a subset Y of edges such that $Y = \delta(X)$ for some $X \subseteq E(G)$.

Assume that G is cubic. A *look* on G is a mapping $\omega : E(G) \to \{-1, 0, 1\}$ such that $\omega(\delta(\{v\})) = 0$, where $\omega(Y) := \sum_{e \in Y} \omega(e)$ for every $Y \subseteq E$. A look ω is *good* if $\omega(D) \neq 1 - |D|$ for every edge-cut D. In particular, if ω is a good look of G and H is the graph obtained from G by removing all the edges in $\omega^{-1}(\{-1\})$, then any cut-edge e of H satisfies $\omega(e) = 1$.

The relevance of good looks is made explicit by the following proposition.

Proposition 3. *Every planar cubic graph with k good looks has at least $k^{1/3}$ perfect matchings.*

Proof. Let ω be a good look of G, and let H_ω be obtained from G by removing the edges in $\omega^{-1}(\{-1\})$ and adding an edge parallel to every edge e with $\omega(e) = 1$. Thus, H_ω is cubic and planar. Moreover, since ω is a good look of G, the graph H_ω has no cut-edge. So H is a planar bridgeless cubic graph and hence it is 3-edge-colourable by the 4-Colour Theorem. Consequently, there exist three disjoint perfect matchings of H_ω. They yield three (non-necessarily) disjoint perfect matchings of G. So each good look ω gives rise to a triplet T_ω of perfect matchings of G.

We assert that the mapping $f : \omega \mapsto T_\omega$ is injective, i.e. two different good looks of G give rise to two different triplets of perfect matchings. To see this, just notice that for every good look ω, every edge e of G is in exactly $1 + \omega(e)$ perfect matchings of T_ω. Now, the fact that f is injective implies that the number of triplets is at least k, and hence the number of perfect matchings of G is at least $k^{1/3}$. □

2 The proof of Theorem 1

The result proved is stronger than Theorem 1. This fact is crucial to deal with the 3- and 2-connected cases.

Let G be a graph and C an even cycle of G. A mapping $\omega_0 : E(C) \to \{-1, 1\}$ such that the edges of C are mapped alternately to -1 and 1 is a *bracelet* on C. A *bracelet of* G is a mapping ω on the edges of G for which there exists an even cycle C of G for which ω restricted to $E(C)$ is a bracelet on C. Then, C is the *supporting cycle* of ω. Given a bracelet ω_0 on C, the *look of the bracelet* ω_0 is the mapping ω defined by $\omega(e) := \omega_0(e)$ if $e \in E(C)$, and $\omega(e) := 0$ otherwise. Note that if ω_0 and ω_1 are two bracelets of G with disjoint supporting cycles, then the sum of their look is a look of G.

Let G be a cubic graph. A *jewel-box* for G is a set $\mathscr{B}$ of bracelets of G such that

- every two members of $\mathscr{B}$ have disjoint supporting cycles; and
- for every subset $W \subseteq \mathscr{B}$, the sum of the looks of the members of W is a good look.

Let $\beta(G)$ be the cardinality of a largest jewel-box of G.

A cubic graph G is *cyclically* 4*-connected* if it is 3-connected, and for every set $X \subset V(G)$ such that both $|X|$ and $V(G) \setminus X$ are at least 2, the size of $\delta(X)$ is at least 4. We prove the following.

Theorem 4. *For every cyclically* 4*-connected cubic graph G,*

$$\beta(G) \geq \frac{|V(G)|}{30976}\,.$$

Note that Theorem 4 implies Theorem 1 by Proposition 3.

We start with the following lemma, which illustrates some standard counting techniques in planar graphs (e.g. the use of the 4-Colour Theorem to ensure the existence of an independent set containing at least a quarter of the vertices).

Lemma 5. *Let G be a simple planar graph and let $A \subseteq V(G)$ be an independent set of G. Set $d := \max\{\deg_G(a) : a \in A\}$. Then, there exist $X \subseteq A$ and $Y \subseteq V(G) \setminus A$ such that*

- $|X| \geq \frac{|A|}{64 \cdot d + 8}$;
- *each member of X is adjacent to at most two members of Y; and*
- *every two members of X are at distance at least* 4 *in $G - Y$.*

Proof. Define Y to be the set of vertices in $V(G) \setminus A$ with at least 10 neighbours in A. We proceed in three steps. Let A_1 be the set of vertices in A with at most two neighbours in Y, and set $A_2 := A \setminus A_1$. We assert that $|A_1| \geq \frac{1}{2} \cdot |A|$. To see this, assume that $A_2 \neq \emptyset$, and so $Y \neq \emptyset$. Hence, $|A \cup Y| \geq 11$. Let H_1 be the bipartite subgraph of G with vertex-set $A \cup Y$ and edge-set composed of all the edges of G between A and Y. Since H_1 is a simple bipartite planar graph on at least 3 vertices, it follows from Euler's Formula that $|E(H_1)| \leq 2 \cdot |V(H_1)| - 4$. On the other hand, $|E(H_1)| \geq 10 \cdot |Y|$ by the definition of Y. Consequently, $10|Y| \leq 2(|A| + |Y|) - 4$ and thus $|Y| \leq \frac{1}{4} \cdot |A|$. Now, let H_2 be the subgraph of H_1 induced by $A_2 \cup Y$. Then $|V(H_2)| \geq 4$ since $A_2 \neq \emptyset$. So, Euler's Formula implies that $|E(H_2)| \leq 2 \cdot (|A_2| + |Y|) - 4$. Moreover, it follows from the definition of A_2 that $|E(H_2)| \geq 3 \cdot |A_2|$. Thus, we deduce that $|A_2| \leq 2 \cdot |Y|$. Therefore, we obtain $|A_2| \leq 2 \cdot \frac{1}{4} \cdot |A|$, and hence $|A_1| \geq \frac{1}{2} \cdot |A|$, as asserted.

It remains to find a subset of A_1 satisfying the third condition (and being large enough), which is done in two steps. First, we define H_3 to be the graph with vertex-set A_1 in which two distinct vertices u and v are adjacent if and only if there exists a path of length 2 from u to v in $G - Y$. We conclude the proof by showing that H_3 contains a sufficiently large independent set. Recall that every vertex of A has degree at most d in G, and every vertex of $V(G) \setminus (A \cup Y)$ has at most 9 neighbours in Y. Thus, it follows that for each vertex $v \in A$, there are at most $8d$ paths of length 2 in $G - Y$ between v and A. In other words, the maximum degree of H_3 is at most $8d$. Hence, H_3 can be properly vertex coloured with $8d + 1$ colours. So, H_3 has an independent set A_3 of size at least $|V(H_3)|/(8d+1) = |A_1|/(8d+1)$. Note that every two vertices of A_3 are at distance at least 3 in $G - Y$.

Now, let G' be the graph obtained from $G - Y$ by contracting every edge with an end-vertex in A_3. Let H_4 be the subgraph of G' induced by A_3. Thus, H_4 has vertex-set A_3, and two distinct vertices of A_3 are adjacent in H_4 if and only if they are at distance 3 in $G - Y$. Moreover, H_4 is planar and simple. By the 4-Colour Theorem, H_4 has an independent set A_4 of size at least $\frac{1}{4} \cdot |A_3|$. Observe that A_4 is an independent set of G contained in A and satisfying the second and third condition of the lemma. Further,

$$|A_4| \geq \frac{1}{4} \cdot |A_3| \geq \frac{|A_1|}{4 \cdot (8d+1)} \geq \frac{|A|}{8 \cdot (8d+1)},$$

which concludes the proof. □

We now prove Theorem 4 in four steps.

Let G be a cyclically 4-connected planar cubic graph with n vertices. We embed G on the sphere Σ (recall that a 3-connected planar graph has a unique embedding on the sphere, and hence a unique dual graph, which is also 3-connected and simple). Let G^* be the dual of G. A *domino* of G is a closed disc $\Delta \subseteq \Sigma$ the boundary of which is a cycle of G containing exactly either one face of G of even length, or two faces of G of odd length.

Lemma 6. *There exists at least $\frac{n}{32}$ pairwise disjoint dominos of G, each having a boundary of length at most* 15*.*

Proof. Let f be the number of faces of G. By Euler's Formula,

$$f = |E(G)| - |V(G)| + 2 = \frac{n}{2} + 2\,.$$

The dual G^* of G is a planar triangulation, which is 4-connected (why?). Hence, Whitney's theorem [6] ensures that G^* is Hamiltonian (this theorem was generalised to all 4-connected planar graphs by Tutte [4]). Let $F_1, f_2, \dots, F_f$ be an enumeration of the faces of G that corresponds to a Hamilton cycle of G^*. Hence, for every integer $i \in \{1, 2, \dots, r\}$, the faces F_i and F_{i+1} of G share an edge (where the subscript is modulo r). Without loss of generality, we may assume that F_f is a face of G of maximum size. Let $k := \lfloor f/2 \rfloor$. The average length of the faces $F_1, F_2, \dots, F_{2k}$ is at most that of all the faces of G, which is less than 6 (since every planar graph is 5-degenerate).

For every $i \in \{1, 2, \dots, k\}$, the closure of one of F_{2i-1}, F_{2i} and $F_{2i-1} \cup F_{2i}$ is a domino Δ_i of G. The length of Δ_i is at most the sum of the lengths of F_{2i-1} and F_{2i} minus 2. Hence, the average length of the dominos $\Delta_1, \Delta_2, \dots, \Delta_k$ is less than 10.

Let us check that at least half of them have length at most 15. Indeed, the minimum length of a domino is 4 (recall that G is simple), so letting x be the number of dominos of length more than 15, we obtain

$$16x + 4(k - x) < 10k\,,$$

i.e. $x < k/2$. Now, say that two dominos are *adjacent* if their boundaries share an edge in G. We let H be the graph defined by this adjacency relation on the at least $k/2$ dominos Δ_i of length less than 16. Note that H is loopless and planar. Thus, the 4-Colour Theorem implies that H has an independent set of size at least

$$\frac{1}{4} \cdot \frac{k}{2} \geq \frac{f-1}{16} > \frac{n}{32}\,,$$

which concludes the proof. □

In the rest, we let A be a set of dominos given by Lemma 6. Let R be the set of all faces not contained in any member of A. Let H be the graph with vertex-set $A \cup R$, in which $\Delta \in A$ and $F \in R$ are adjacent if and only if the boundaries of Δ and F share an edge, and two distinct faces f and f' in R are adjacent if their boundaries share an edge. So, H is simple (since G is 3-connected) and planar. Moreover, A induces an independent set of H composed of vertices of degree at most 15. As a result, Lemma 5 implies the existence of two sets $X \subseteq A$ and $Y \subseteq R$ such that

- $|X| \geq |A|/968 \geq n/30976$;
- each member of X is adjacent to at most two members of Y; and
- every two members of X are at distance at least 4 in $G - Y$.

Let us write $X = \{\Delta_1, \ldots, \Delta_k\}$ with $k \geq n/30976$. We want to use X to construct a jewel-box of G. For $i \in \{1, \ldots, k\}$, let C_i be the cycle of G forming the boundary of Δ_i. There are two bracelets on C_i, and we choose one as follows.

If Δ_i is adjacent in H to at most one member of Y, then choose any bracelet ω_i of C_i. Otherwise, let F_1 and F_2 be the two neighbours of Δ_i that belong to Y. If F_1 or F_2 shares a unique edge e with Δ_i, then choose the bracelet ω_i on C_i such that $\omega_i(e) = 1$. Otherwise, i.e. if both F_1 and F_2 share more than one edge with Δ_i, then choose any bracelet ω_i on C_i.

It remains to show that $\{\omega_1, \ldots, \omega_k\}$ is a jewel-box of G. Choose a set $W \subseteq \{1, \ldots, k\}$, and let ω be the sum of the looks of all bracelets ω_i with $i \in W$. We need to show that ω is a good look. As we noted earlier, ω is a look since the supporting cycles of the looks ω_i are pairwise disjoint. Hence, we now check that it is good, i.e. for every edge-cut D of G, it holds that $\omega(D) \neq 1 - |D|$.

Suppose on the contrary that D is an edge-cut of G such that $\omega(D) = 1 - |D|$. We choose D such that $|D|$ is minimal subject to this property. Note that there exists

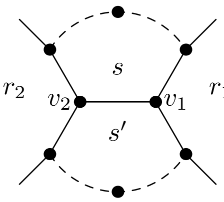

Figure 1: The domino Δ_j is composed of the two faces s and s', each being of odd length.

an edge $f \in D$ such that $\omega(f) = 0$, and all the others edges of D are mapped to -1 by ω. First, we observe that D is then a minimal edge-cut of G. Indeed, if $D' \subset D$ is also an edge-cut of G, then so is $D \setminus D'$. Hence, we may assume that $f \in D'$. Therefore, D' is an edge-cut of G such that $\omega(D') = 1 - |D'|$ and $|D'| < |D|$, which contradicts the minimality of $|D|$.

Since D is a minimal edge-cut, there is a cycle C in G^* such that $E(C) = D$ (from now on, we identify the edges of G and G^* in the natural way). Let S be the set of faces of G not contained in R, i.e. the set of faces contained in some domino $\Delta \in A$. For $i \in \{-1, 0, 1\}$, we set $\Phi_i := \omega^{-1}(\{i\})$. Hence, Φ_{-1}, Φ_0 and Φ_1 form a partition of the edges of G, exactly one edge of D belongs to Φ_0, all the other belonging to Φ_{-1}.

Lemma 7. *Let $s \in V(C) \cap S$ and let r_1 and r_2 be its two neighbours in C. Then one of r_1 and r_2 does not belong to Y.*

Proof. Suppose on the contrary that both r_1 and r_2 are in Y, and hence in R. Let Δ be the (unique) domino of A that includes s (it exists since $s \in S$). Let e_i be the edge (of C) between s and r_i, for $i \in \{1, 2\}$. Since $D = E(C)$, at least one of e_1 and e_2 belongs to Φ_{-1}. Consequently, $\Delta \in X$. Let us write $\Delta = \Delta_j$ with $j \in W$. So both e_1 and e_2 belong to the boundary of Δ_j, and hence $\omega_j(e_1) = -1 = \omega_j(e_2)$. Therefore, it follows from the definition of ω_j that each of r_1 and r_2 shares two edges with the boundary of Δ_j. Since G is 3-connected, this implies that Δ_j contains two (odd) faces s and s', which share an edge v_1v_2 drawn in the interior of Δ_j in the embedding of G (See Figure 1). Further, both r_1 and r_2 share an edge with both s and s'. Because G is cyclically-4-connected, neither r_1 nor r_2 is incident with both v_1 and v_2. Thus, we may assume that r_i is incident with v_i (and not with v_{3-1}) for $i \in \{1, 2\}$. But then, the fact that $\omega_j(e_1) = \omega_j(e_2)$ contradicts that the face s had odd lenght (see Figure 1). This concludes the proof. □

Lemma 8. *Let e be the unique edge of C in Φ_0. In G^*, either both ends of e are in R or both are in S.*

Proof. Suppose on the contrary that $e = rs$ with $r \in R$ and $s \in S$. Let r' be the second neighbour of s in C. Since $\omega_j(r's) = -1$ and $s \in S$, we deduce that s is contained in a domino Δ_j with $j \in W$. So every edge of the boundary of Δ_j belongs to $\Phi_{-1} \cup \Phi_1$, which contradicts that $e \in \Phi_0$. □

We prove a last lemma before concluding the proof of Theorem 1.

Lemma 9. *Assume that $e' = ss'$ is an edge of G^* with $s, s' \in S$. If both s and s' belong to C, then they are adjacent in C.*

Proof. Let P_1 and P_2 be the two paths between s and s' on C. In one of them, say P_1, every edge is in Φ_{-1}. Let e be the edge of G corresponding to the edge e' of G^*. In the embedding of G, the edge e is drawn in the interior of a member of A. Therefore, $e \in \Phi_0$. So, the cycle C_1 of G^* obtained by adding e to P_1 satisfies $\omega(E(C_1)) = 1 - |E(C_1)|$. Consequently, it follows from the minimality of D that $|E(C_1)| = |E(C)|$, i.e. P_2 is the single edge e. □

We now conclude the proof of Theorem 4. Recall that G^* is simple and 3-connected (because G is 3-connected), so C has length has least 3. Moreover, every edge in Φ_{-1} has an end in R and the other in S (in G^*). Consequently, Lemma 8 implies that the length of C is odd. Moreover, C has a unique edge e with both ends in R or both ends in S, and $e \in \Phi_0$. Let us write $|E(C)| = 2t + 1$ for some positive integer t.

First, suppose that $= 1$. Then, the three faces of G corresponding to the vertices of C are pairwise adjacent. Since G is cyclically-4-edge connected, these three faces share a common vertex v. Now, two edges incident with v belongs to Φ_{-1}, a contradiction.

Assume now that $t \geq 2$. We consider two cases regarding whether C has two consecutive vertices that belong to S. Suppose that $C = s_0, r_1, s_1, r_2, s_2, \ldots, r_t, s_t$. For each $i \in \{1, 2, \ldots, t\}$, there is a domino $\Delta \in A$ such that $s_i \in \Delta$. Since, $r_i s_i \in \Phi_{-1}$, it follows that $\Delta = \Delta_{j(i)}$ for some $j(i) \in W$. Lemma 9 implies that $j(i) \neq j(i')$ if $i \neq i'$. So we may assume that $j(i) = i$ for every $i \in \{1, 2, \ldots, t\}$. Note that Δ_t is the closure of s_0 and s_t. Now, in H the vertex r_t is adjacent to two vertices of X, namely Δ_{t-1} and Δ_t. Thus, $r_t \in Y$. Similarly, $r_{t-1} \in Y$. Thus, s_1 contradicts Lemma 7.

It remains to deal with the case where C has two consecutive vertices that belong to R. Let us write $C = r_0, s_1, r_1, s_2, r_2, \ldots, s_t, r_t$. As before, we can assume that for each $i \in \{1, 2, \ldots, t\}$ the domino Δ_i contains s_i. If $t \geq 3$, then (in H) both r_1 and r_2 belong to Y. Hence, s_2 contradicts Lemma 7. So $t = 2$; in this case, observe that one of r_0 and r_2 belongs to Y, since $\Delta_2, r_0, r_2, \Delta_1$ is a path of H of length 2 between Δ_1 and Δ_2, two members of X. As a result, s_1 or s_2 contradicts Lemma 7 since $r_1 \in Y$. This contradiction concludes the proof of Theorem 4.

3 Concluding Words

What about the general case? Deducing the general case from Theorem 1 is not easy. Actually, Chudnovsky and Seymour point out that they need to use the stronger statement of Theorem 4, rather than just that of Theorem 1. They proceed in two steps: first they extend Theorem 4 to 3-connected planar cubic graphs, and then to general planar cubic graphs (obtaining a smaller constant each time).

It seems hard to proceed by induction, though it is very tempting. Rather, the authors decompose the graph using a *set* of 3-edge-cuts. To do so, they introduce a nice tool, called a *cut-decomposition*. Let us end by stating the definition.

Let G be a graph. A *cut-decomposition* of G is a pair (T, Φ) where

- T is a tree with at least one edge;
- Φ is a map from $V(G)$ to $V(T)$; and
- for each $t \in V(T)$ of degree at most 2, there exists a vertex v of G such that $\Phi(v) = t$.

Let $e \in E(T)$ and let T_1 and T_2 be the two components of $T - e$. Then, $\delta(\Phi^{-1}(V(T_1)) = \delta(\Phi^{-1}(V(T_2))$ is a cut of G.

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